[lex159] Jaynes-Cummings model I

The Hamiltonian of the Jaynes-Cummings model of a two-level atom interacting with a single quantized mode of radiation field [lln27] in scaled form can be rendered as follows:

$$\mathcal{H} = S_z + a^\dagger a + \Lambda(S_+ a + S_- a^\dagger).$$

The Hilbert space is spanned by the orthonormal basis vectors, $|m,n\rangle$; $m = 0, 1$, $n = 0, 1, 2, \ldots$ with the following actions of the relevant operators:

$$S_z|m,n\rangle = \big(\tfrac{1}{2} - m\big)|m,n\rangle, \quad S_+|m,n\rangle = \sqrt{m(1-m+1)}\,|m-1,n\rangle,$$
$$S_-|m,n\rangle = \sqrt{(1-m)(m+1)}\,|m+1,n\rangle,$$
$$a^\dagger|m,n\rangle = \sqrt{n+1}\,|m,n+1\rangle, \quad a|m,n\rangle = \sqrt{n}\,|m,n-1\rangle.$$

(a) Determine the matrix elements for the four terms of $\mathcal{H}$,

$$\langle mn|S_z|n'm'\rangle, \quad \langle mn|a^\dagger a|n'm'\rangle, \quad \langle mn|S_+ a|n'm'\rangle, \quad \langle mn|S_- a^\dagger|n'm'\rangle.$$

(b) Show by inspection that the eigenvectors with even parity are

$$|\psi_{1,0}\rangle = |1,0\rangle, \quad |\psi_{1,n}\rangle = \frac{1}{\sqrt{2}}\big[|1,n\rangle + |0,n-1\rangle\big] \quad : \; n = 1, 2, \ldots$$

and the eigenvectors with odd parity are

$$|\psi_{0,n}\rangle = \frac{1}{\sqrt{2}}\big[|1,n+1\rangle - |0,n\rangle\big] \quad : \; n = 0, 1, 2, \ldots$$

with respective eigenvalues,

$$E_{1,n} = \hbar\omega\left(n - \frac{1}{2}\right) + \hbar g_0\sqrt{n}, \quad E_{0,n} = \hbar\omega\left(n + \frac{1}{2}\right) - \hbar g_0\sqrt{n+1},$$

(c) Identify the ground state and the first pair of eigenstates whose energy levels are subject to interaction-mediated splitting.

Solution: