[lex157] Einstein coefficient for hydrogen $1s \to 2p$ transition

The electronic wave functions for the $1s$ and $2p$ states (with $m = 0$) of the hydrogen atom are real,

$$\psi_1(r,\theta) = \pi^{-1/2} a_0^{-3/2} e^{-r/a_0}, \quad \psi_2(r,\theta) = \pi^{-1/2}(2a_0)^{-5/2} r \cos\theta \, e^{-r/2a_0},$$

where a_0 is the Bohr radius. Transitions with $\Delta m = 0$ are not prohibited if the light is polarized in z-direction.
(a) Check the normalization of both wave functions.
(b) Calculate the electric-dipole matrix element,

$$\mu_{12} \doteq -e \int d^3r \, \psi_1^*(\mathbf{r}) z \psi_2(\mathbf{r}),$$

for the $1s \to 2p$ transition as a function of a_0 and the elementary charge e.
(c) Calculate the numerical value (in SI units) of the Einstein coefficient, using the expression,

$$B_{12} = \frac{\pi \mu_{12}^2}{3\epsilon_0 \hbar^2},$$

determined in [lln27].

Solution: