[lex155] Coherent state of quantum harmonic oscillator II

The pure one-parameter quantum state,

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle,$$

where α is the complex-valued parameter, is known as a coherent state of the quantum harmonic oscillator used for modeling a stream of photons [lln26]. Consider the displacement operator,

$$D = e^{\imath(p\mathcal{Q}-x\mathcal{P})/\hbar}, \quad \mathcal{Q} = \sqrt{\frac{\hbar}{2m\omega}}\big(a^\dagger + a\big), \quad \mathcal{P} = \imath\sqrt{\frac{m\hbar\omega}{2}}\big(a^\dagger - a\big),$$

where x, p is a point in the phase plane and $\mathcal{Q}, \mathcal{P}$ are position and momentum operators expressed in terms of ladder operators [lam5].
(a) Show that the translation operator expressed by ladder operators becomes,

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \quad \alpha = \sqrt{\frac{\omega m}{2\hbar}}\,x + \imath\sqrt{\frac{1}{2\hbar\omega m}}\,p = \Re[\alpha] + \imath\Im[\alpha].$$

(b) Show that the displacement operator generates (non-stationary) coherent states $|\alpha\rangle$ from the (stationary) ground state $|0\rangle$ as follows: $D(\alpha)|0\rangle = |\alpha\rangle$.

Hint: Use the operator relation, $e^{A+B+[A,B]/2} = e^A e^B$, which is valid if the operator $[A, B]$ commutes with operators A and B.

Solution: