[lex152] Erlang distributions II

Consider the (continuous) Erlang distribution for arbitrary integer-valued index k,

$$f^{(k)}(t) \doteq \frac{(k/\tau)^k t^{k-1}}{(k-1)!}\, e^{-kt/\tau}.$$

Prepare a Mathematica notebook or equivalent to carry out the following list of tasks:
(a) Show that function $f^{(k)}(t)$ is normalized.
(b) Show that its mean is equal to τ, and its variance equal to τ^2/k.
(c) Infer from $f^{(k)}(t)$ the discrete probability distribution,

$$P_n^{(k)}(t) = \left[\sum_{m=0}^{k-1} \frac{(kt/\tau)^{kn+m}}{(kn+m)!}\right] e^{-kt/\tau} = \frac{\Gamma(kn+k, kt/\tau)}{\Gamma(kn+k)} - \frac{\Gamma(kn, kt/\tau)}{\Gamma(kn)},$$

via integration and the use of a recursion relation as follows:

$$P_0^{(k)}(t) = \int_t^\infty dt'\, f^{(k)}(t'), \quad P_n^{(k)}(t) = \int_0^t dt'\, f^{(k)}(t') P_{n-1}^{(k)}(t-t').$$

(d) Show that this (discrete) probability distribution is normalized as well.
(e) Plot $P_0^{(k)}(t)$ as a function of t/τ for increasing values of k and infer the function $P_0^{(\infty)}(t)$.

Solution: