[lex134] Longitudinal and transverse current densities

In some applications, notably those that work with the Coulomb gauge condition, it is useful to split the current density into an (irrotational) longitudinal part and a (solenoidal) transverse part:

$$\mathbf{J} = \mathbf{J}_\mathrm{L} + \mathbf{J}_\mathrm{T}, \quad \nabla \times \mathbf{J}_\mathrm{L} = 0, \quad \nabla \cdot \mathbf{J}_\mathrm{T} = 0.$$

Show that the two parts can be separated by the following integral expressions:

$$\mathbf{J}_\mathrm{L}(\mathbf{x}) = -\frac{1}{4\pi}\nabla \int d^3x' \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \quad \mathbf{J}_\mathrm{T}(\mathbf{x}) = \frac{1}{4\pi}\nabla \times \nabla \times \int d^3x' \frac{J(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$$

Hint: Start from the identity $\nabla \times (\nabla \times \mathbf{J}) = \nabla(\nabla \cdot \mathbf{J}) - \nabla^2 \mathbf{J}$ and use integrations by part.

Solution: