[lex125] Conducting hemispheres at opposite electric potential

A thin spherical shell of radius a centered a the origin of the coordinate system consists of two conducting hemispheres electrically insulated from each other and oppositely charged to electric potential $\pm\Phi_0$ at $z = \pm 0$, respectively. (a) Use the Green's function established in [lln21],

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\int d\Omega' \Phi(a,\theta',\phi')\,\frac{a(x^2-a^2)}{\left[x^2+a^2-2ax\cos\gamma\right]^{3/2}} \quad : \; x > a,$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$, to infer an explicit integral expression for the electric potential in the space outside the sphere.
(b) Evaluate the resulting integral expression for field points on the z-axis.
(c) Expand the integrand of the expression obtained for part (a) in powers of a/x and evaluate the integral for the leading term and the first correction.

Solution: