[lex123] Hyperfine coupling

The task is to extract from the Hamiltonian of an orbiting electron in the magnetic field generated by a nuclear spin,

$$\mathcal{H} = \frac{1}{2m_\mathrm{e}}\left(\mathbf{p}+e\mathbf{A}\right)^2 + 2\mu_\mathrm{B}\mathbf{S}\cdot(\nabla\times\mathbf{A}) + \Phi(r), \quad \mathbf{A} = \frac{\mu_0}{4\pi r^3}\,\boldsymbol{\mu}\times\mathbf{r}, \quad \boldsymbol{\mu} = g_I\mu_\mathrm{N}\mathbf{I},$$

the part which describes the hyperfine coupling in two steps as follows:

$$\mathcal{H}' \doteq \mathcal{H} - \frac{p^2}{2m_\mathrm{e}} - \Phi(r) - \frac{e^2}{2m_\mathrm{e}}\mathbf{A}^2 = \frac{\mu_0 g_I\mu_\mathrm{B}\mu_\mathrm{N}}{2\pi}\left[(\mathbf{S}\cdot\nabla)(\mathbf{I}\cdot\nabla)\frac{1}{r} - (\mathbf{S}\cdot\mathbf{I})\nabla^2\frac{1}{r} + \frac{\mathbf{L}\cdot\mathbf{I}}{r^3}\right],$$

$$= \frac{\mu_0 g_I\mu_\mathrm{B}\mu_\mathrm{N}}{2\pi}\,\mathbf{I}\cdot\left[\frac{\mathbf{L}-\mathbf{S}+3(\mathbf{S}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}}{r^3} + \frac{8\pi}{3}\,\mathbf{S}\,\delta(\mathbf{r})\right].$$

Hint: The Fermi contact interaction, which accounts for the fact that the nucleus is not strictly a point, adds a term $\frac{1}{3}\sum_{ij} S_j I_i \nabla^2(1/r)\delta_{ij}$ to $(\mathbf{S}\cdot\nabla)(\mathbf{I}\cdot\nabla)(1/r)$.

Solution: