[lex120] Electrostatic interaction energy of two point charges

The energy density of the electrostatic field generated by two point charges is

$$\begin{aligned} w(\mathbf{x}) &= \frac{1}{2}\epsilon_0 \left[\frac{q_1}{4\pi\epsilon_0} \frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} + \frac{q_2}{4\pi\epsilon_0} \frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_2|^3} \right]^2 \\ &= \frac{1}{32\pi^2\epsilon_0} \left[\frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + 2q_1q_2 \frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3} \right]. \end{aligned}$$

The first two terms contribute to the (divergent) electrostatic self-energy and the last term to the interaction energy. Show step by step that the integral of the last term yields the familar expression of interaction energy for a pair of point charges.

$$U_{\text{int}} = \frac{q_1q_2}{16\pi^2\epsilon_0} \int d^3x \frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3} = \frac{1}{4\pi\epsilon_0} \frac{q_1q_1}{|\mathbf{x}_1-\mathbf{x}_2|}.$$

Solution: