[lex116] Poynting vector and radiation power of electric dipole

(a) Given the electric and magnetic radiation fields,

$$\mathbf{E}(\mathbf{x},t)_{\mathrm{rad}} = \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right) - \left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right], \quad \mathbf{B}(\mathbf{x},t)_{\mathrm{rad}} = -\frac{\mu_0}{4\pi r c}\hat{\mathbf{r}}\times\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}, \quad t_r \doteq t - \frac{r}{c},$$

worked out in [lex222] and [lex114] for the radiation of an electric dipole $\mathbf{p}(t)$, show that the Poynting vector in the radiation zone becomes

$$\mathbf{S}(\mathbf{x},t)_{\mathrm{rad}} \doteq \frac{1}{\mu_0}\mathbf{E}(\mathbf{x},t)_{\mathrm{rad}}\times\mathbf{B}(\mathbf{x},t)_{\mathrm{rad}} = \frac{\mu_0/c}{(4\pi r)^2}\left[\left(\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right)^2 - \left(\hat{\mathbf{r}}\cdot\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right)^2\right]\hat{\mathbf{r}},$$

representing the energy per unit time and and unit area radiated in its direction.
(b) Calculate the radiation power of the electric dipole $\mathbf{p}(t)$ from the integral of the vector $\mathbf{S}$ over a spherical surface at radius r,

$$P(r,t) = \oint d\Omega\, r^2\, \hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{x},t)_{\mathrm{rad}}.$$

This quantity is independent of r as dictated by energy conservation.

Solution: