[lex111] Green's function of operator $\nabla^2 + k^2$

The function $G(\mathbf{x}-\mathbf{x}')$ is the Green's function of the operator $\nabla^2 + k^2$ if it satisfies the equation,

$$-(\nabla^2 + k^2)G(\mathbf{x}-\mathbf{x}') = 4\pi\delta(\mathbf{x}-\mathbf{x}'). \tag{1}$$

Show that the Green's function depends on the distance variable, $r \doteq |\mathbf{x}-\mathbf{x}'|$, and has the form,

$$G(r) = \frac{e^{\imath kr}}{r}. \tag{2}$$

Hint: Use spherical coordinates and full rotational symmetry. Show first that the left-hand side of (1) with (2) vanishes for $r \neq 0$. The integral of the right-hand side over the volume of a sphere of radius $R > 0$ is 4π. Show that the same is the case for the left-hand side.

Solution: