[lex108] Helmholtz equation for wave guide II: TM modes

Consider TM modes in a wave guide with a cross section of arbitrary shape $\mathcal{C}$. From [lex104] we know that the ansatz,

$$\mathbf{B}(\mathbf{x},t) = -\nabla \times \left[\psi(x,y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{k}}\right],$$

satisfies the wave equation if the function $\psi(x,y)$ satisfies the Helmholtz equation,

$$\nabla^2\psi = -\gamma^2\psi, \quad \gamma^2 = +\frac{\omega^2}{c^2} - k^2.$$

(a) Use Ampère's law to show that the associated magnetic field is

$$\mathbf{E}(\mathbf{x},t) = \frac{c^2 k}{\omega}\left[\nabla\psi(x,y) - \imath\frac{\gamma^2}{k}\psi(x,y)\,\hat{\mathbf{k}}\right] e^{\imath(kz-\omega t)}.$$

(b) Show that the boundary conditions, $\mathbf{B}_\perp = 0$ and $\mathbf{E}_\parallel = 0$ are encoded in $\psi = 0$ for points on $\mathcal{C}$.

Solution: