[lex107] Helmholtz equation for wave guide I: TE modes

Consider TE modes in a wave guide with a cross section of arbitrary shape $\mathcal{C}$. In [lex104] we have shown that the ansatz,

$$\mathbf{E}(\mathbf{x},t) = \nabla \times \Big[-\psi(x,y) e^{\imath(kz-\omega t)}\, \hat{\mathbf{k}} \Big],$$

satisfies the wave equation if the function $\psi(x,y)$ satisfies the Helmholtz equation,

$$\nabla^2 \psi = -\gamma^2 \psi, \quad \gamma^2 = +\frac{\omega^2}{c^2} - k^2.$$

(a) Use Faraday's law to show that the associated magnetic field is

$$\mathbf{B}(\mathbf{x},t) = \frac{k}{\omega} \Big[-\nabla\psi(x,y) + \imath \frac{\gamma^2}{k} \psi\, \hat{\mathbf{k}} \Big] e^{\imath(kz-\omega t)}.$$

(b) Show that the boundary conditions, $\mathbf{B}_\perp = 0$ and $\mathbf{E}_\parallel = 0$ are encoded in $\hat{\mathbf{n}} \cdot \nabla\psi = 0$ for all points on $\mathcal{C}$.

Solution: