[lex104] TE mode in rectangular wave guide

(a) Show that the ansatz,

$$\mathbf{E}(\mathbf{x},t) = \nabla \times \Big[-\psi(x,y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{k}}\Big] = \left[\frac{\partial}{\partial x}\psi(x,y)\,\hat{\mathbf{j}} - \frac{\partial}{\partial y}\psi(x,y)\,\hat{\mathbf{i}}\right] e^{\imath(kz-\omega t)},$$

for the electric field of TE modes in a rectangular wave guide (see [lln18]) satisfies the wave equation, $c^2\nabla^2\mathbf{E} - \partial^2\mathbf{E}/\partial t^2 = 0$, if the scalar function $\psi(x,y)$ satisfies the Helmholtz equation,

$$\nabla^2\psi = -\gamma^2\psi, \quad \gamma^2 = +\frac{\omega^2}{c^2} - k^2.$$

(b) Given the TE(m,n) solution for the electric field,

$$\mathbf{E}(\mathbf{x},t) = \Psi_0 e^{\imath(kz-\omega t)}\left[\frac{n\pi}{b}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\hat{\mathbf{i}} - \frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{j}}\right],$$

Use Faraday's law to show that the associated magnetic field is

$$\mathbf{B}(\mathbf{x},t) = \frac{k}{\omega}\Psi_0 e^{\imath(kz-\omega t)}\left[\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{i}} \right.$$
$$\left. + \frac{n\pi}{b}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\hat{\mathbf{j}} + \imath\frac{\gamma^2}{k}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{k}}\right].$$

Solution: