Electrostatics III [lln7]

Uniqueness theorem for Laplace equation:

Laplace equation: $\nabla^2\Phi(\mathbf{x}) = 0$.

Boundary: Surface S that either fully encloses a finite region of space V or separates an infinite region from other regions. Part of S may be at infinity.

Dirichlet boundary conditions: $\Phi(\mathbf{x})$ known on S.

Neumann boundary conditions: $\hat{\mathbf{n}} \cdot \nabla\Phi(\mathbf{x})$ known on S.

Uniqueness theorem: Any solution $\Phi(\mathbf{x})$ of the Laplace equation with a full set of boundary conditions (Dirichlet or Neumann or mixed) is unique.

A proof of the uniqueness theorem is sketched in the following:

- ▷ Can two solutions $\Phi_1(\mathbf{x})$, $\Phi_2(\mathbf{x})$ be distinct? The theorem says no.
- ▷ We begin by defining a vector function with product structure:

$$\mathbf{F}(\mathbf{x}) \doteq \underbrace{[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})]}_{g(\mathbf{x})}\,\underbrace{\nabla[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})]}_{\mathbf{G}(\mathbf{x})}.$$

- ▷ Next we invoke Gauss's theorem,

$$\int_V d^3x\, \nabla \cdot \mathbf{F}(\mathbf{x}) = \oint_S d\mathbf{a} \cdot \mathbf{F}(\mathbf{x}).$$

for the vector field $\mathbf{F}(\mathbf{x})$.

- ▷ Since both solutions $\Phi_1(\mathbf{x})$, $\Phi_2(\mathbf{x})$ are identical at the boundaries, the surface integral vanishes and, therefore, the volume integral as well:

$$\int_V d^3x\, \nabla \cdot \mathbf{F}(\mathbf{x}) = \int_S da[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})]\hat{\mathbf{n}} \cdot \nabla[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})] = 0. \quad (1)$$

- ▷ Given that the construction of $\mathbf{F}(\mathbf{x})$, we can apply the following mathematical identity (well established in vector analysis) to it:

$$\nabla \cdot (g\mathbf{G}) = \nabla g \cdot \mathbf{G} + g\nabla \cdot \mathbf{G}.$$

- ▷ Carrying out this task and recalling that $\Phi_1(\mathbf{x})$, $\Phi_2(\mathbf{x})$ both solve the Laplace equation we obtain

$$\nabla \cdot \mathbf{F}(\mathbf{x}) = \nabla[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})] \cdot \nabla[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})] + [\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})]\,\underbrace{\nabla^2[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})]}_{0}. \quad (2)$$

▷ Using the result of (2) in the volume integral of (1) yields:

$$\int_V d^3x \big\{\nabla[\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})]\big\}^2 = 0.$$

▷ The integrand is non-negative. Therefore, integral can only vanish if the integrand vanishes identically.

▷ Two functions whose gradients are identical can only differe by a constant.

$$\Phi_1(\mathbf{x}) = \Phi_2(\mathbf{x}) + \text{const.}$$

▷ The boundary conditions demand that this constant is zero.

▷ We conclude that the solution of the Laplace equation is unique:

Keep in mind that the uniqueness theorem as proven here does not make any claims regarding the uniqueness of charge configurations on the surfaces of conductors that eneter as boundary conditions.

Separable solutions in Cartesian coordinates:

Cartesian coordinates are the preferred choice for electrostatic situations with flat and rectangular boundaries.

Laplace equation in Cartesian coordinates:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \Phi(x, y, z) = 0. \tag{3}$$

The methodology explored here aims to reduce the PDE (3) into ODEs by means of a product ansatz for the solution:

$$\Phi(x, y, z) \doteq X(x)Y(y)Z(z). \tag{4}$$

Each term in (3) acts on a different factor of (4):

$$\frac{d^2X(x)}{dx^2}\, Y(z)Z(z) + X(x)\, \frac{d^2Y(y)}{dy^2}\, Z(z) + X(x)Y(y)\, \frac{d^2Z(z)}{dz^2} = 0.$$

Division by $\Phi(x, y, z)$ makes each term a function of a different variable:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0.$$

An equation of this type can only be satisfied if each term is a constant and the three constants add up to zero:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \kappa_1^2, \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \kappa_2^2, \quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = \kappa_3^2, \tag{5}$$

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 0. \tag{6}$$

Writing the constants as squares of constants κ_i is a choice of convenience. The condition (6) can only be satisfied if any nonzero and real κ_i is accompanied by at least one imaginary κ_j.

The types of general solution of the ODEs (5) can be classified as follows:

- ▷ $\kappa_1^2 = 0$: $X(x) = ax + b$,
- ▷ $\kappa_1^2 > 0$: $X(x) = a\, e^{\kappa_1 x} + b\, e^{-\kappa_1 x} = c \sinh(\kappa_1 x) + d \cosh(\kappa_1 x)$,
- ▷ $\kappa_1^2 < 0$: $X(x) = a\, e^{\imath k_1 x} + b\, e^{-\imath k_1 x} = c \sin(k_1 x) + d \cos(k_1 x)$, $\quad \kappa_1 = \imath k_1$.

Note the higher symmetry of the hyperbolic and trigonometric functions. The same classification obtains for the solutions $Y(y)$ and $Z(z)$.

Permissible combinations which satisfy (6)

- ▷ $\kappa_1^2 = \kappa_2^2 = \kappa_3^2 = 0$.
- ▷ $\kappa_i^2 = -\kappa_j^2 \neq 0, \quad \kappa_k^2 = 0$.
- ▷ $\kappa_i^2 > 0, \quad \kappa_j^2 > 0, \quad \kappa_k^2 = -\kappa_i^2 - \kappa_j^2 < 0$.
- ▷ $\kappa_i^2 < 0, \quad \kappa_j^2 < 0, \quad \kappa_k^2 = -\kappa_i^2 - \kappa_j^2 > 0$,

The assignment of the κ_i to specific (Cartesian) coordinates is dictated by the boundary conditions of a given application.

The second set of κ_i applies (not exclusively) to situations which are effectively two-dimensional due to a translational symmetry along one of the coordinate axes.

In general, the physically relevant solution is a linear combination of product solutions of the types identified, often in the form of expansions in sets of orthogonal functions.

Applications to situations with rectangular boundary conditions are found in the exercises. [lex20][lex21]

[lex126][lex127]

Separable solutions in spherical coordinates:

In the previous module we have already explored solutions of the Laplace equation in spherical coordinates.

[lln6] Here we pursue this goal more systematically for cases with the full spherical symmetry broken but azimuthal symmetry still intact.

The analysis of the general case (without any kind of rotational symmetry) is significantly more complex and discussed in the additional materials. [lam13]

Electric potential for applications with azimuthal symmetry: $\Phi(r, \theta)$.

[gmd2-A] Laplace equation (for this case):

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \, \frac{\partial \Phi}{\partial \theta}\right) = 0.$$

Product ansatz: $\Phi(r, \theta) \doteq R(r)\Theta(\theta)$.

Substitute product ansatz for $\Phi(r, \theta)$ into Laplace equation and divide $\Phi(r, \theta)$. These operations separate the two variables:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2 \frac{d}{dr}\, R(r)\right) + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta \, \frac{d}{d\theta}\, \Theta(\theta)\right) = 0.$$

The first term is a function of r and the second a function θ. The equation can only hold if they are constants, equal in magnitude and opposite in sign.

Judicious choice of constant: $l(l+1) \quad : \; l = 0, 1, 2, \ldots$

The Laplace equation (PDE) is thus reduced to ODEs for radial function $R(r)$ and angular function $\Theta(\theta)$:

$$\frac{d}{dr}\left(r^2 \, \frac{dR}{dr}\right) - l(l+1)R = 0, \tag{7}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \, \frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0. \tag{8}$$

The general solution of ODE (7) is readily found:[1]

$$R(r) = A\, r^l + B\, r^{-(l+1)}.$$

It has two integration constants as expected. Furthermore, it shares the separation constant l as a parameter with the solution of (8).

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[1]Note that alternative factorizations are in common use, for example, the following: $\Phi(r, \theta) = [U(r)/r]\Theta(\theta) \quad \Rightarrow \; U(r) = Ar^{(l+1)} + Br^{-l}$.

The ODE (8) can be transformed into into the (well documented) Legendre equation as sketched in the following:

[lam16]

- $\triangleright$ $u \doteq \cos\theta \quad \Rightarrow \quad \dfrac{du}{d\theta} = -\sin\theta = -\sqrt{1-u^2}, \quad P(u) \doteq \Theta(\theta).$
- $\triangleright$ $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta} = \dfrac{1}{\sin\theta}\dfrac{d}{du}\dfrac{du}{d\theta} = -\dfrac{d}{du}.$
- $\triangleright$ $\sin\theta\,\dfrac{d\Theta}{d\theta} = \sin\theta\,\dfrac{dP}{du}\dfrac{du}{d\theta} = -(1-u^2)\dfrac{dP}{du}.$
- $\triangleright$ Legendre equation: $\dfrac{d}{du}\left[(1-u^2)\,\dfrac{dP}{du}\right] + l(l+1)P = 0.$

Legendre polynomials $P_l(u)$ are regular solutions of Legendre equation.

- $\triangleright$ Variable range: $-1 \leq u \leq 1.$
- $\triangleright$ $P_0(u) = 1, \quad P_1(u) = u, \quad P_2(u) = \frac{1}{2}(3u^2-1), \quad \ldots$
- $\triangleright$ $P_l(u)$ is symmetric (antisymmetric) for even (odd) l.
- $\triangleright$ The $P_l(u)$ are a set of orthogonal polynomials.

Expansion of azimuthally symmetric electric potential:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l\, r^l + B_l\, r^{-(l+1)}\right] P_l(\cos\theta). \tag{9}$$

Physical meaning of specific terms:

- $\triangleright$ $l = 0 \quad \Rightarrow \Phi(r) = A_0 + \dfrac{B_0}{r}.$

  A_0: arbitrary constant,

  B_0/r: potential of spherically symmetric charge distribution inside radius r.
- $\triangleright$ $l = 1 \quad \Rightarrow \Phi(r,\theta) = A_1 r\cos\theta + \dfrac{B_1\cos\theta}{r^2}.$

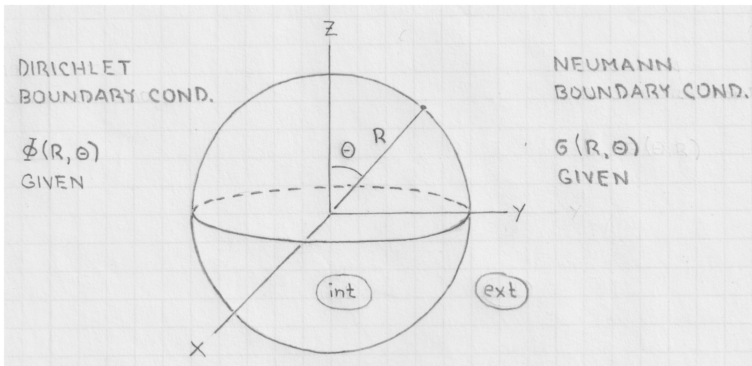
  $A_1 r\cos\theta$: potential generated by uniform electric field in z-direction,

  $B_1\cos\theta/r^2$: potential generated by electric dipole oriented in z-direction.

Consider solutions,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l\, r^l + B_l\, r^{-(l+1)}\right] P_l(\cos\theta).$$

with boundary conditions specified on sphere of radius R.



- ▷ Non-divergent Φ at $r < R$ (interior region) requires that $B_l^{(\mathrm{int})} = 0$.
- ▷ Vanishing Φ for $r \to \infty$ (exterior region) requires that $A_l^{(\mathrm{ext})} = 0$.
- ▷ Continuity of Φ at $r = R$ requires matching expansion coefficients in both regions. This requirement can be satisfied by the condition,

$$A_l^{(\mathrm{int})} R^l = B_l^{(\mathrm{ext})} R^{-(l+1)} \doteq a_l.$$

Series expansions of potential in interior and exterior regions:

$$\Phi_{\mathrm{int}}(r,\theta) = \sum_{l=0}^{\infty} a_l \left(\frac{r}{R}\right)^l P_l(\cos\theta) \quad : \ r \leq R, \tag{10}$$

$$\Phi_{\mathrm{ext}}(r,\theta) = \sum_{l=0}^{\infty} a_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta) \quad : \ r \geq R. \tag{11}$$

Dirichlet boundary conditions at $r = R$: $\Phi(R,\theta) = \Phi_0(\theta)$.

Their implementation uses the expansions (10), (11) and the orthogonality
[lam16] of Legendre polynomials: :

$$\int_0^{\pi} d\theta \sin\theta\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\,\delta_{ll'},$$

The expansion coefficients follow directly:

$$\Phi_0(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos\theta),$$

$$\int_0^\pi d\theta \sin\theta P_{l'}(\cos\theta)\Phi_0(\theta) = \sum_{l=0}^{\infty} a_l \underbrace{\int_0^\pi d\theta \sin\theta P_{l'}(\cos\theta) P_l(\cos\theta)}_{\frac{2}{2l+1}\delta_{ll'}}.$$

$$\Rightarrow \quad a_l = \frac{2l+1}{2} \int_0^\pi d\theta \sin\theta \, P_l(\cos\theta)\Phi_0(\theta).$$

Neumann boundary conditions at $r = R$: $\quad -\left[\frac{\partial \Phi_{\text{ext}}}{\partial r} - \frac{\partial \Phi_{\text{int}}}{\partial r}\right]_{r=R} = \frac{\sigma_0(\theta)}{\epsilon_0}.$

Their implementation again uses the expansions (10), (11), and the orthogonality of Legendre polynomials:

$$\left.\frac{\partial \Phi_{\text{ext}}}{\partial r}\right|_R = -\frac{1}{R}\sum_{l=0}^{\infty}(l+1)a_l P_l(\cos\theta), \qquad \left.\frac{\partial \Phi_{\text{int}}}{\partial r}\right|_R = \frac{1}{R}\sum_{l=0}^{\infty} l a_l P_l(\cos\theta).$$

$$\Rightarrow \quad \frac{\sigma_0(\theta)}{\epsilon_0} = -\left[\frac{\partial \Phi_{\text{ext}}}{\partial r} - \frac{\partial \Phi_{\text{int}}}{\partial r}\right]_{r=R} = \frac{1}{R}\sum_{l=0}^{\infty}(2l+1)a_l P_l(\cos\theta),$$

$$\int_0^\pi d\theta \sin\theta \, \frac{\sigma_0(\theta)}{\epsilon_0} \, P_{l'}(\cos\theta) = \frac{1}{R}\sum_{l=0}^{\infty}(2l+1)a_l \underbrace{\int_0^\pi d\theta P_{l'}(\cos\theta) P_l(\cos\theta)}_{\frac{2}{2l+1}\delta_{ll'}}.$$

$$\Rightarrow \quad a_l = \frac{R}{2\epsilon_0}\int_0^\pi d\theta \sin\theta \, P_l(\cos\theta)\sigma_0(\theta).$$

The case of surface charge density, $\sigma_0(\theta) = \sigma_0 \cos\theta$, on the spherical shell represents an electric dipole. The exterior potential and field are those of an electric dipole. The interior electric field is uniform.

Two applications are found in the exercises. Legendre polynomials also fature prominently in an off-center expansion of the Coulomb potential. [lex22][lex23] [lex24]

Separable solutions in cylindrical coordinates:

Here again we elaborate on the more playful tinkering with the Laplace equation in cylindrical coordinates undertaken in the previous module.

[lln6] We only consider situations with continuous translational symmetry along the z-axis. The (more complex) analysis of the general case is discussed in [lam14] the additional materials.

Electric potentials for applications with translational symmetry: $\Phi(r, \phi)$.

Laplace equation (for this case):

$$r \frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right) + \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \tag{12}$$

[gmd2-A]

Product ansatz: $\Phi(r, \phi) \doteq R(r)\Psi(\phi)$.

Substitute product ansatz for $\Phi(r, \phi)$ into Laplace equation and divide $\Phi(r, \phi)$. These operations separate the two variables:

$$\frac{r}{R}\frac{d}{dr}\left(r \frac{dR}{dr}\right) + \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = 0.$$

The first term is a function of r and the second a function ϕ. The equation can only hold if they are constants, equal in magnitude and opposite in sign.

Judicious choice of separation constant: ν^2

The separation reduces the PDE to a pair of 2^{nd}-order ODEs.

General solution for $\nu = 0$:

- $\triangleright$ $\displaystyle \frac{r}{R}\frac{d}{dr}\left(r \frac{dR}{dr}\right) = 0 \quad \Rightarrow \; R(r) = a \ln r + b,$
- $\triangleright$ $\displaystyle \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} = 0 \quad \Rightarrow \; \Psi(\phi) = c\,\phi + d.$

General solution for $\nu \neq 0$:

- $\triangleright$ $\displaystyle \frac{r}{R}\frac{d}{dr}\left(r \frac{dR}{dr}\right) - \nu^2 = 0 \quad \Rightarrow \; R(r) = ar^{\nu} + br^{-\nu},$
- $\triangleright$ $\displaystyle \frac{1}{\Psi}\frac{d^2\Psi}{d\phi^2} + \nu^2 = 0 \quad \Rightarrow \; \Psi(\phi) = c\cos(\nu\phi) + d\sin(\nu\phi).$

Each general solution features two integration constants and, for $\nu \neq 0$, the value of the separation constant.

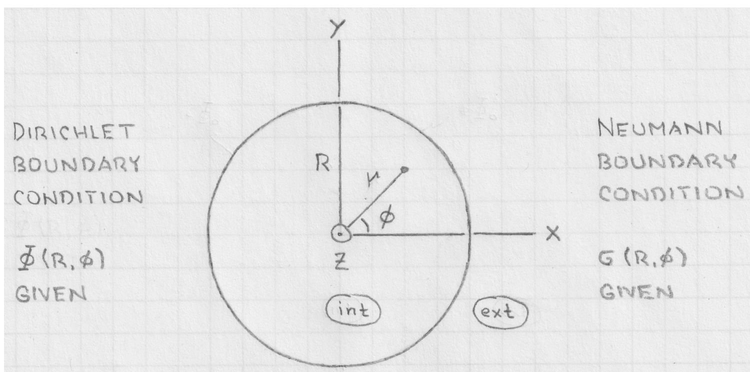
Azimuthically periodic solutions of the Laplace equation (12) require that the separation constant is integer-valued: $\nu = n \in \mathbb{Z}$.

$$\Phi(r,\phi) = a\ln r + b + \sum_{n=1}^{\infty}\big[a_n r^n + b_n r^{-n}\big]\big[c_n\cos(n\phi) + d_n\sin(n\phi)\big]. \quad (13)$$

The first two terms of (13) represent the the electric potential of a uniformly
[lln5] charged rod of infinite length as discussed in an earlier module:

$$\Phi(r) = a\ln r + b = a\ln\left(\frac{r}{r_0}\right). \quad (14)$$

For applications of the remaining terms in (13), consider situations of zero net charge overall. Boundary conditions are specified on a cylinder radius R. The potential must vanish at infinite radius.



The set of coefficients is reduced as we impose the following three conditions:

- ▷ Non-divergent Φ at $r < R$ (interior region) requires that $b_n^{(\text{int})} = 0$.
- ▷ Vanishing Φ for $r \to \infty$ (exterior region) requires that $a_n^{(\text{ext})} = 0$.
- ▷ Continuity of Φ at $r = R$ requires matching expansion coefficients in both regions: $a_n^{(\text{int})} R^n = b_n^{(\text{ext})} R^{-n}$.

Series expansions of potential in interior and exterior regions:

$$\Phi_{\text{int}}(r,\phi) = \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\Big[c_n\cos(n\phi) + d_n\sin(n\phi)\Big] \quad : \ r \leq R, \quad (15)$$

$$\Phi_{\text{ext}}(r,\phi) = \sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n\Big[c_n\cos(n\phi) + d_n\sin(n\phi)\Big] \quad : \ r \geq R. \quad (16)$$

Dirichlet boundary conditions at $r = R$: $\Phi(R, \phi) = \Phi_0(\phi)$.

Their implementation uses the expansions (15), (16) evaluated at $r = R$, and the orthogonality of sines and cosines:

$$\Phi_0(\theta) = \sum_{n=1}^{\infty} \Big[c_n \cos(n\phi) + d_n \sin(n\phi) \Big], \tag{lex25}$$

$$\int_0^{2\pi} d\phi \cos(n\phi)\cos(n'\phi) = \int_0^{2\pi} d\phi \sin(n\phi)\sin(n'\phi) = \pi\,\delta_{nn'},$$

$$\int_0^{2\pi} d\phi \cos(n\phi)\sin(n'\phi) = 0,$$

$$\Rightarrow \quad c_n = \frac{1}{\pi}\int_0^{2\pi} d\phi \cos(n\phi)\Phi_0(\phi), \quad d_n = \frac{1}{\pi}\int_0^{2\pi} d\phi \sin(n\phi)\Phi_0(\phi).$$

Neumann boundary conditions at $r = R$: $-\left[\frac{\partial \Phi_{\rm ext}}{\partial r} - \frac{\partial \Phi_{\rm int}}{\partial r}\right]_{r=R} = \frac{\sigma_0(\phi)}{\epsilon_0}$.

Their implementation uses partial derivatives of the expansions (15), (16) evaluated at $r = R$, and the orthogonality of sines and cosines:

$$\frac{\sigma_0(\theta)}{\epsilon_0} = \frac{1}{R}\sum_{n=1}^{\infty} 2n \Big[c_n \cos(n\phi) + d_n \sin(n\phi) \Big],$$

$$\int_0^{2\pi} d\phi \cos(n\phi)\cos(n'\phi) = \int_0^{2\pi} d\phi \sin(n\phi)\sin(n'\phi) = \pi\,\delta_{nn'},$$

$$\int_0^{2\pi} d\phi \cos(n\phi)\sin(n'\phi) = 0,$$

$$\Rightarrow \quad c_n = \frac{R}{2n\pi\epsilon_0}\int_0^{2\pi} d\phi \cos(n\phi)\sigma_0(\phi), \quad d_n = \frac{R}{2n\pi\epsilon_0}\int_0^{2\pi} d\phi \sin(n\phi)\sigma_0(\phi).$$

Special case for which the series expansion terminates naturally:

- ▷ Neumann boundary condition: $\sigma_0(\phi) = \sigma_0 \cos\phi \quad : \ r = R$.
- ▷ Sole non-vanishing expansion coefficient:

$$c_1 = \frac{R\sigma_0}{2\pi\epsilon_0} \underbrace{\int_0^{2\pi} d\phi \cos^2\phi}_{\pi} = \frac{R\sigma_0}{2\epsilon_0}.$$

- ▷ Electric potential in interior and exterior regions:

$$\Phi_{\text{int}}(r,\phi) = \frac{\sigma_0}{2\epsilon_0}\, r\cos\phi = \frac{\sigma_0}{2\epsilon_0}\, x, \quad \Phi_{\text{ext}}(r,\phi) = \frac{\sigma_0 R^2}{2\epsilon_0 r}\, \cos\phi.$$

- ▷ Uniform electric field in interior region:

$$\mathbf{E}_{\text{int}} = -\nabla\Phi_{\text{int}} = -\frac{\sigma_0}{2\epsilon_0}\,\hat{\mathrm{i}}.$$

A nonzero net charge on the cylinder requires that we add the term (14) with logarithmic r-dependence to the potential.

Solutions of the Laplace equation from conjugate functions:

Here we consider situations with continuous translational symmetry perpendicular to the xy-plane and use elements of complex analysis.

Electric potential: $\Phi(x, y)$.

Laplace equation: $\dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial^2\Phi}{\partial y^2} = 0$.

Cartesian coordinates: x, y.

Complex coordinate and its conjugate: $z = x + \imath y, \quad \bar{z} = x - \imath y$.

Consider two differentiable real functions: $g(x, y), \quad h(x, y)$.

From any pair of such functions a complex function can be constructed:

$$F(z, \bar{z}) = g(x, y) + \imath h(x, y) = \Re[F(z, \bar{z})] + \imath\,\Im[F(z, \bar{z})].$$

A complex functions is analytic (complex differentiable) if its real part $g(x, y)$
[gmd7-A] and and its imaginary part $h(x, y)$ satisfy the Cauchy-Riemann conditions,

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}, \quad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}. \tag{17}$$

The functions g and h are then named conjugate functions and the analytic function can be expressed as $F(z)$ (a function of z alone).

In consequence, the real and imaginary parts of $F(z)$ are harmonic functions. They satisfy the Laplace equation separately:

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0, \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0. \tag{18}$$

Significance for electrostatics:

- Pick an analytic function $F(z)$ such that either $\Re[F(z)]$ or $\Im[F(z)]$ satisfies the Dirichlet boundary conditions of an electrostatic situation which calls for a solution $\Phi(x, y)$ of the Laplace equation.
- Given (18), the conjugate function which does satisfy the boundary conditions is the solution sought: $\Phi(x, y) = g(x, y)$ or $\Phi(x, y) = h(x, y)$.
- The conjugate function which does not satisfy the boundary conditions has a physically significant role as well: the lines $h(x, y) = \text{const}$ or
[gmd7-A] $g(x, y) = \text{const}$ are electrostatic field lines.

Application to charged plane surface:

Consider a plane surface with charge density σ on yz-plane as shown.

Analytic function: $F(z) = cz = cx + \imath cy$.

Real and imaginary parts: $g = cx, \quad h = cy$.

Cauchy-Riemann conditions: $\dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y} = c, \quad \dfrac{\partial g}{\partial y} = -\dfrac{\partial h}{\partial x} = 0.$

Harmonic functions: $\dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} = 0, \quad \dfrac{\partial^2 h}{\partial x^2} + \dfrac{\partial^2 h}{\partial y^2} = 0.$

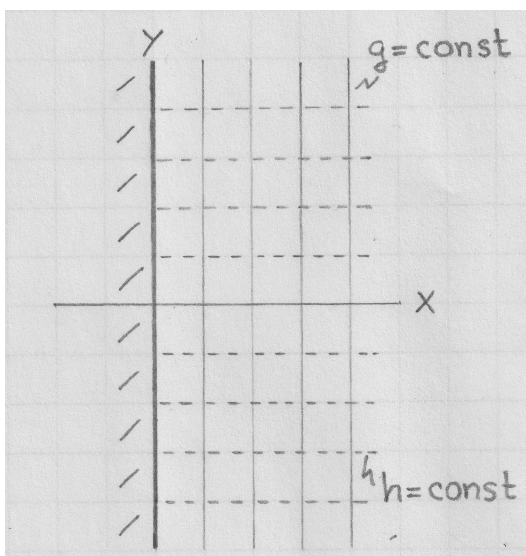
Dirichlet boundary condition: $\Phi(0, y) = \text{const}$.

Equipotential lines: $g = \text{const} \quad \Rightarrow \; x = \text{const} \quad$ (solid lines).

Field lines: $h = \text{const} \quad \Rightarrow \; y = \text{const} \quad$ (dashed lines).

Electric field: $\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{i}} = -\nabla\Phi.$

Electric potential: $\Phi = -\dfrac{\sigma}{\epsilon_0}\,x.$



Further applications which employ Cartesian coordinates are found in the exercises. [lex26][lex27] [lex28]

Application to charged cylindrical surface:

Consider a cylindrical surface of radius r_0 centered around z-axis with charge density σ as shown in cross section.

Analytic function: $F(z) = c \ln z = c \ln r + \imath c\phi, \quad z = re^{\imath\phi}$.

Real part: $g = c \ln r, \quad r = \sqrt{x^2 + y^2}$.

Imaginary part: $h = c\phi, \quad \phi = \arctan \dfrac{y}{x}$.

Checking Cauchy-Riemann conditions ensures that $g(x, y)$ nad $h(x, y)$ are harmonic functions (solutions of the Laplace equation):

$$\frac{\partial g}{\partial x} = \frac{dg}{dr}\frac{\partial r}{\partial x} = \frac{cx}{x^2 + y^2}, \quad \frac{\partial h}{\partial y} = \frac{dh}{d\phi}\frac{\partial \phi}{\partial y} = \frac{cx}{x^2 + y^2}.$$

$$\frac{\partial g}{\partial y} = \frac{dg}{dr}\frac{\partial r}{\partial y} = \frac{cy}{x^2 + y^2}, \quad \frac{\partial h}{\partial x} = \frac{dh}{d\phi}\frac{\partial \phi}{\partial x} = -\frac{cy}{x^2 + y^2}.$$

Equipotential lines: $g = \text{const} \quad \Rightarrow r = \text{const}$ (concentric circles).

Field lines: $h = \text{const} \quad \Rightarrow \phi = \text{const}$ (radial lines).

Electric potential: $\Phi = \Phi_0 \ln(r/r_0)$.

Electric field: $\mathbf{E} = -\nabla\Phi = -\dfrac{\Phi_0}{r}\hat{\mathbf{r}} = \dfrac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}} = \dfrac{\sigma r_0}{\epsilon_0 r}\hat{\mathbf{r}}, \quad \lambda \doteq 2\pi r_0 \sigma$.

Field strength at $r = r_0$: $E_0 = \dfrac{\sigma}{\epsilon_0} = -\dfrac{\Phi_0}{r_0} \quad \Rightarrow \Phi_0 = -\dfrac{\sigma r_0}{\epsilon_0}$.