# Electrostatics II [lln6]

### Electrical insulators and conductors:

Solid materials consist of strongly coupled atoms. Atoms consist of positively charged nuclei and negatively charged electronic shells. Electrons may be exchanged in ionic bonds or shared in valence bonds between neighboring atoms. Conduction electrons are shared by all atoms in metallic bonds.

Relevant properties of solid materials:

- *Insulators:* Insulators can be electrically charged via surface treatment. Techniques exist to produce volume charge densities. All charges in insulators are frozen in place. The charge distribution is static.
- *Conductors:* Conduction electrons are free to move through the material. While drifting, they dissipate energy in collisions. Conductors left in isolation will reach equilibrium with a static charge distribution.
- *Semiconductors:* The binding of some electrons is sufficiently weak that they can hop between atoms.
- *Superconductors:* Interaction-mediated pairing of electrons enables collision-free motion (it's a long story for a different course).

The focus here is on electrical conductors. The analysis uses a length scale on which a continuum description is adequate. Charge distributions are averaged over interatomic distances.

### Electrostatic properties of conductors:

A conductor has reached equilibrium when the mobile charges experience no net force on the length scale in use.

This has consequences.

#1 The electric field $\mathbf{E}(\mathbf{x})$ inside the conducting material vanishes.

#2 The electric potential $\Phi(\mathbf{x})$ is the same everywhere in a conductor.

#3 The charge density $\rho(\mathbf{x})$ vanishes inside the conducting material.

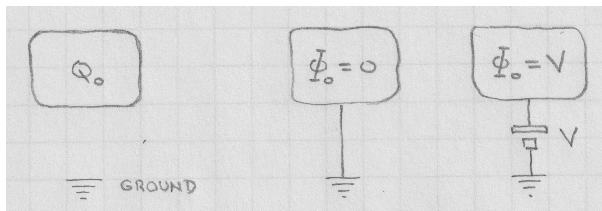
#4 All excess charge $Q$ on a charged conductor is located at the surface. The surface charge density $\sigma$ may vary between points.

#2 follows from #1 and the relation $\mathbf{E} = -\nabla\Phi$. #3 is a consequence of Gauss's law: any supposed interior charge could be wrapped into a Gaussian surface; the absence of a field implies the absence of flux, which, in turn, implies the absence of charge inside.

**Specification of conductor at equilibrium:**

Two alternatives;

- ▷ The charge on the conductor is kept at a fixed value: $Q_0 = \text{const}$. This conductor is isolated from other conductors.
- ▷ The conductor is kept at a fixed potential: $\Phi_0 = \text{const}$. This conductor is either grounded or connected to the ground by a voltage source.



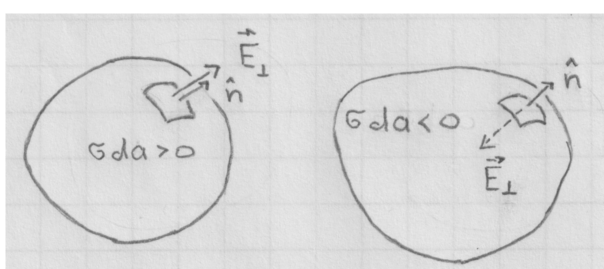

A conductor is grounded when in contact with a conductor of much bigger size. A charge transfer that is significant for the small conductor remains insignificant for the combined conductor and, therefore, does not produce a change in potential.

**Electric field at the surface of a conductor (just outside):**

- ▷ $\mathbf{E}_{\|} = 0$ (tangential components),
- ▷ $\mathbf{E}_{\perp} = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{n}}$ (normal component).

Consequences of facts established earlier: (i) $\mathbf{E} = 0$ just inside; (ii) $\mathbf{E}_{\|}$ is
[lln5] continuous across surface; (iii) $\mathbf{E}_{\perp}$ has a discontinuity $\sigma/\epsilon_0$ across surface.

Note: unit vector $\hat{\mathbf{n}}$ points toward outside; charge density $\sigma$ may vary between points on the surface and change sign in the process.



The uniformity of the charge density on some conducting surfaces can be justified by symmetry:

- A charged plane conducting sheet of infinite extension has $\sigma = \text{const}$ due to symmetry under translations in two directions.
- A charged conducting sphere has $\sigma = \text{const}$ due to symmetry under rotations by polar and azimuthal angles.
- A charged conducting cylinder of infinite length has $\sigma = \text{const}$ due to symmetry under translations along its axis and rotations about its axis.

The three surfaces are described by keeping one Cartesian, spherical, or cylindrical coordinate fixed and varying the other two over their full range. [lex207][lex208]

In all three cases a vector tangential to the surface has two components in symmetry directions. This makes all positions on the surface equivalent for scalar quantities such as $\sigma$. [lex209] [lex210][lex211]

Predicting the charge density on the surface of a charged conductor of lower symmetry is challenging, in general.

One notable result in this category is a conductor of ellipsoidal shape,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The charge density for points on this ellipsoidal surface is[1]

$$\sigma(x, y, z) = \frac{Q}{4\pi abc}\left[\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right]^{-1/2}.$$

Investigating how the distribution of surface shifts on this object as its shape change is instructive and will pursued in selected exercises. [lex212]

The anchor point of such investigations is the spherical limit, $a = b = c = R$, for which we recover the familiar result, [lex213][lex214]

$$\sigma \;\to\; \frac{Q}{4\pi R^3}\left[\frac{R^2}{R^4}\right]^{-1/2} = \frac{Q}{4\pi R^2}$$

Other limiting cases pertain to charged thin straight wires of finite length and charged conducting disks of finite radius.

[1] W. R. Smythe, *Static and dynamic electricity*, McGraw-Hill, New York 1968, p.124, cited in Griffiths 2023.

**Electrostatic pressure:**

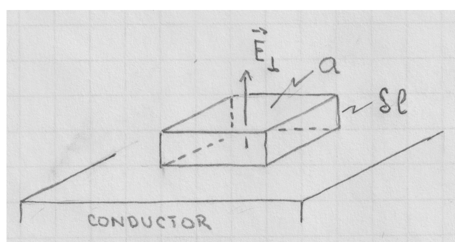
Ultimate source: Coulomb repulsion of excess charge on surface.
Immediate source: normal electric field $\mathbf{E}_\perp$ exerts outward force $\mathbf{F}_\perp$.

- ▷ Electric field outside surface: $E_\perp = \dfrac{\sigma}{\epsilon_0}$.
- ▷ Potential energy of thin layer outside surface: $\delta U = \frac{1}{2}\epsilon_0 E_\perp^2 a\delta l$.
- ▷ Work done by force $F_\perp$ when surface is moved outward:

$$F_\perp \delta l = \delta U = \frac{1}{2}\epsilon_0 E_\perp^2 a\delta l = \frac{\sigma^2}{2\epsilon_0}\, a\delta l.$$

- ▷ Electrostatic pressure (force per area): $\dfrac{\mathbf{F}_\perp}{a} = \dfrac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{n}}$.

Note: electrostatic pressure always pushes toward the outside of a conducting surface, irrespective of whether it is charged positively or negatively.




**Existence and uniqueness of electrostatic equilibrium:**

Required specifications for a collection of insulators and conductors:

- ▷ *Insulators:* Charge distribution $\rho_l(\mathbf{x})$ throughout the space occupied by each insulator.
- ▷ *Conductors:* Either the (uniform) electric potential $\Phi_k$ or the total charge $Q_k$ on each conductor.

If only insulators are present, the electrostatic field configuration is the result
[lln5] of an integral. The existence and uniqueness of the electrostatic equilibrium
is evident. All electric charges are fixed in space.

In the presence of conductors, the electrostatic equilibrium is facilitated by the inevitable energy dissipation associated with motion of charges in conductors. This equilibration mechanism alone does not guarantee uniqueness.
[lln7] A uniqueness theorem is required and does exist.

**Boundary value problem:**

Finding the electrostatic potential $\Phi(\mathbf{x})$ for a configuration of conductors amounts to the solution of the Laplace equation,[2]

$$\nabla^2\Phi(\mathbf{x}) = 0,$$

throughout the space outside any conducting material and subject to boundary conditions at all surfaces.

▷ *Dirichlet* boundary conditions: The potential $\Phi(\mathbf{x})$ is given on all surfaces. It is known to be a constant on each surface.

▷ *Neumann* boundary conditions: The charge density $\sigma(\mathbf{x}) = -\epsilon_0\hat{\mathbf{n}}\cdot\nabla\Phi(\mathbf{x})$ is given on all surfaces. It is known to be, in general, nonuniform. The unit vector $\hat{\mathbf{n}}$ is directed normal to the surface away from the conductor.

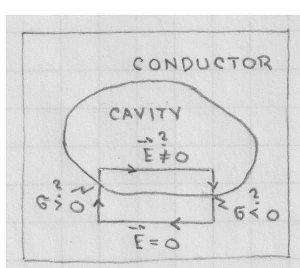
**Electric field in empty cavity:**

In electrostatic equilibrium, the surface charge density $\sigma$ on the wall of an empty cavity and the electric field $\mathbf{E}$ inside the cavity vanish identically.

This result can be arrived at via different chains of reasoning:

▷ The cavity wall is an equipotential surface, $\Phi = \Phi_0$, constituting the complete boundary for cavity the solution of the Laplace equation. The solution, $\Phi(\mathbf{x}) = \Phi_0 = \text{const}$ is a solution of the Laplace equation. The uniqueness theorem eliminates alternatives. Hence $\mathbf{E} = -\nabla\Phi_0 = 0$.

▷ A Gaussian surface embedded in the conductor and surrounding the cavity has zero electric flux through it, hence zero charge inside. If the surface charge density were only zero on average, it would have regions of positive $\sigma$ and regions of negative $\sigma$. A loop could be constructed, which connects two such regions through the cavity and the remainder through the conductor. The loop integral, $\mathbf{E}\cdot d\mathbf{l}$, would be nonzero in contradiction to the irrotational nature, $\nabla\times\mathbf{E} = 0$, of the electric field.




[2]The Laplace equation is a special case of the Poisson equation introduced in [lln5].

**Capacitor:**

Charged conductors are devices for storing energy. A typical device consists of two oppositely charged conductors near each other. The charges and potentials are $+Q$, $-Q$, and $\Phi_1$, $\Phi_2$, respectively.

[lln5] Earlier we derived two expressions for the electrostatic potential energy:

$$U = \frac{1}{2}\int d^3x \rho(\mathbf{x})\Phi(\mathbf{x}) = \frac{1}{2}\epsilon_0 \int d^3x |\mathbf{E}(\mathbf{x})|^2.$$

The second expression tells us that all energy is in the electrical field, i.e. outside the conductors. The first expression leads to integrals over the surfaces $S_1$ and $S_2$ of the two conductors, the only locations of exess charge.

Since the potential is constant across each surface, the integrals are, effectively, over the surface charge densities:

$$\begin{aligned} U &= \frac{1}{2}\int_{S_1} d^2x \sigma_1(\mathbf{x})\Phi_1(\mathbf{x}) + \frac{1}{2}\int_{S_2} d^2x \sigma_2(\mathbf{x})\Phi_2(\mathbf{x}) \\ &= \frac{1}{2}\Phi_1 \underbrace{\int_{S_1} d^2x \sigma_1(\mathbf{x})}_{+Q} + \frac{1}{2}\Phi_2 \underbrace{\int_{S_2} d^2x \sigma_2(\mathbf{x})}_{-Q} = \frac{1}{2}Q\Phi_1 - \frac{1}{2}Q\Phi_2 = \frac{1}{2}QV. \end{aligned}$$

Potential difference (voltage): $V \doteq \Phi_1 - \Phi_2$.

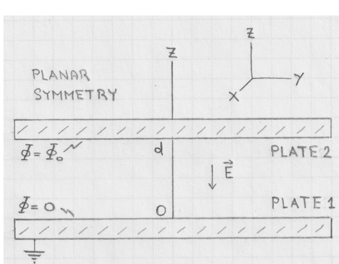
Definition of device property *capacitance*: $C \doteq \dfrac{Q}{V}$ $\quad$ [F] $\doteq$ [C/V].

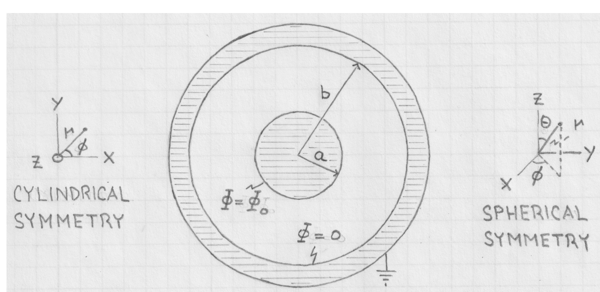
Alternative expressions for energy stored on capacitor:

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}.$$

[lex10] Common geometries: planar, cylindrical, and spherical.

[lex11][lex12]



**Method of images for induced charges:**

We have seen that induced charges on the surface of conductors are only uniformly spread if specific symmetry conditions are satisfied or approached (spheres, long cylinders or wires, large plane sheets).

Nonuniform charge densities are readily induced on conducting surfaces of any symmetry. A charged conducting sheet placed next to an uncharged conducting sphere induces nonuniform charge density on both objects.

When point charges are positioned next to a conducting surface with no prior induced charge, a nonuniform surface charge density $\sigma(\mathbf{x})$ is being induced. It can be determined by the method of images.

The key requirement is the placement of image point charges such that the conducting surface becomes an equipotential surface, $\Phi(\mathbf{x}) = \text{const}$, for the point charge and its images.

- ▷ Electric potential of all point charges in the configuration including image charges:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_k \frac{q_k}{|\mathbf{x} - \mathbf{x}_k|}.$$

- ▷ Electric field associated with $\Phi(\mathbf{x})$: $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$.
- ▷ The vector $\mathbf{E}(\mathbf{x})$ is, by construction, perpendicular to the equipotential surface, $\Phi(\mathbf{x}) = \text{const}$.
- ▷ The surface of the conductor is an equipotential surface.
- ▷ The surface charge density is related to the local electric field via

$$|\sigma(\mathbf{x})| = \epsilon_0 |\mathbf{E}(\mathbf{x})|.$$

- ▷ If the electric field $\mathbf{E}(\mathbf{x})$ is directed away from (toward) the surface, then surface charge density $\sigma$ is positive (negative).

[lex13][lex14]
[lex15][lex16]
[lex18][lex124]

This method works for more general charge configurations positioned near conducting surfaces of various shapes. The placement of the image charges are dictated, to a large extent, by symmetry considerations. Some symmetries are less obvious than others.

**Solutions in search of a problem:**

Solutions of the Laplace equation, $\nabla^2\Phi(\mathbf{x}) = 0$, that satisfy particular symmetry conditions are readily constructed. What electrostatic configurations do they represent? Here we list six candidate cases for applications.

*Cartesian coordinates:* $\Phi(\mathbf{x}) = \Phi(x, y, z)$.

- ▷ Case #1 with planar symmetry.
  - One relevant variable: $\Phi(\mathbf{x}) = \Phi(x)$.
  - Laplace equation: $\dfrac{d^2\Phi}{dx^2} = 0$.
  - General solution: $\Phi(x) = a + bx$.

- ▷ Case #2 with broken planar symmetry.
  - Two relevant variables: $\Phi(\mathbf{x}) = \Phi(x, y)$.
  - Laplace equation: $\dfrac{d^2\Phi}{dx^2} + \dfrac{d^2\Phi}{dy^2} = 0$.
  - 1-parameter solution: $\Phi(x, y) = a(x^2 - y^2)$.

*Spherical coordinates:* $\Phi(\mathbf{x}) = \Phi(r, \theta, \phi)$.

- ▷ Case #3 with spherical symmetry.
  - One relevant coordinate: $\Phi(\mathbf{x}) = \Phi(r)$.
  - Laplace equation: $\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d\Phi}{dr}\right) = 0$.
  - General solution: $\Phi(r) = \dfrac{a}{r} + b$.

- ▷ Case #4 with broken spherical symmetry.
  - Two relevant variables: $\Phi(\mathbf{x}) = \Phi(r, \theta)$.
  - Laplace equation: $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\Phi}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\,\dfrac{\partial\Phi}{\partial\theta}\right) = 0$.
  - 4-parameter solution: $\Phi(r, \theta) = \dfrac{a}{r} + b + \dfrac{c\cos\theta}{r^2} + d\,r\cos\theta$.

*Cylindrical coordinates:* $\Phi(\mathbf{x}) = \Phi(r, \phi, z)$.

▷ Case #5 with cylindrical symmetry.

- One relevant variable: $\Phi(\mathbf{x}) = \Phi(r)$.
- Laplace equation: $\displaystyle\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0$.
- General solution: $\displaystyle\Phi(r) = a \ln\frac{r}{b}$.

▷ Case #6 with broken cylindrical symmetry.

- Two relevant variables: $\Phi(\mathbf{x}) = \Phi(r, \phi)$.
- Laplace equation: $\displaystyle\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$.
- 4-parameter solution: $\displaystyle\Phi(r, \phi) = a \ln\frac{r}{b} + \frac{c\cos\phi}{r} + d\,r\cos\phi$.

In cases #1, #3, and #5 the Laplace equation is a $2^{\text{nd}}$-order ODE. The general solution has two parameters. Particular solutions require two boundary conditions, which determine the parameters.

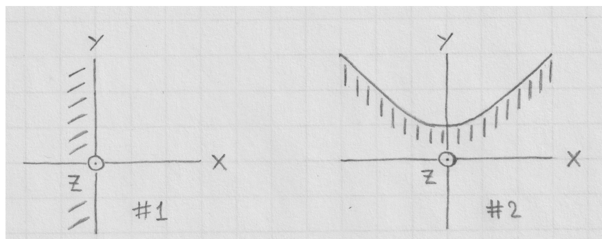
In cases #2, #4, and #6 the Laplace equation is a $2^{\text{nd}}$-order PDE. The concept of general solutions is largely inapplicable. All solutions found have one or several parameters.

Parametric solutions become particular solutions when all parameters are uniquley determined the boundary conditions of specific physical situations.

**Applications for the six cases:**

▷ Case #1: *Plane surface of conductor.*

- Conductor at $x < 0$.
- Surface charge density: $\sigma(0, y, z) = \sigma = \text{const.}$
- Electric potential: $\Phi(\mathbf{x}) = \Phi_0 - \dfrac{\sigma}{\epsilon_0} x \quad : \; x > 0.$
- Electric field: $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{i}} \quad : \; x > 0.$

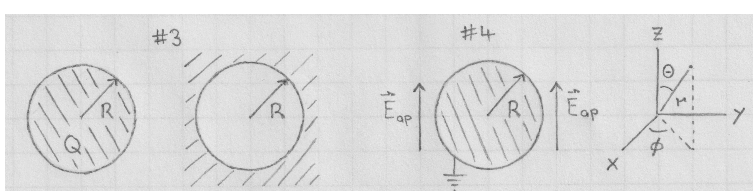



[lex43] ▷ Case #2: *Conducting hyperbolic trough.*

▷ Case #3: *Conducting spherical surface.*

Conductor at $r < R$ (conducting sphere).

- Surface charge density: $\sigma(R) = \sigma = \text{const.}$
- Total charge on surface: $Q = 4\pi R^2 \sigma$.
- Electric potential: $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r} \quad : \; r > R.$
- Electric field: $\mathbf{E}(\mathbf{x}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\,\hat{\mathbf{r}} \overset{r\to R}{\rightsquigarrow} \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{r}}.$

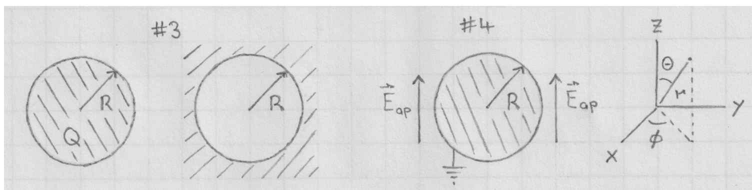


Conductor at $r > R$ (spherical cavity).

- Electric potential $\Phi(R) = \Phi_0 = \text{const}$ on boundary.
- $\Phi(\mathbf{x}) = \Phi_0 = \text{const}$ for $|\mathbf{x}| < R$ is the unique solution.
- Electric field: $\mathbf{E}(\mathbf{x}) = 0$ throughout cavity.
- Surface charge density: $\sigma = 0$ everywhere on cavity surface.

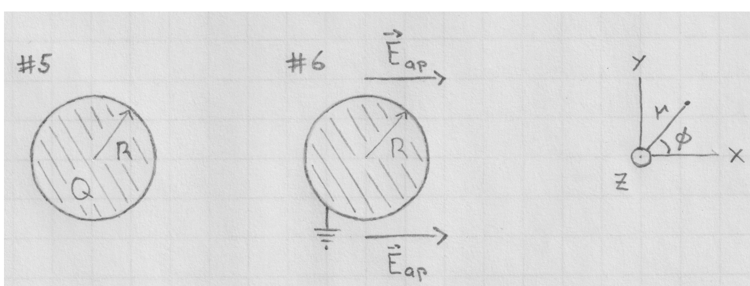
▷ Case #4: *Conducting sphere in uniform electric field.*

[lex17]




▷ Case #5: *Conducting cylindrical surface.*

- Conductor at $r < R$.
- Surface charge density: $\sigma(R) = \sigma = \text{const}$.
- Line charge density: $\lambda(z) = 2\pi R\sigma = \text{const}$.
- Electric potential: $\Phi(\mathbf{x}) = \dfrac{\lambda}{2\pi\epsilon_0} \ln \dfrac{R}{r} \quad : \ r > R.$
- Electric field: $\mathbf{E}(\mathbf{x}) = \dfrac{1}{2\pi\epsilon_0} \dfrac{\lambda}{r} \hat{\mathbf{r}} \quad \overset{r\to R}{\rightsquigarrow} \quad \dfrac{\sigma}{\epsilon_0} \hat{\mathbf{r}} \quad : \ r > R.$




[lex19] ▷ Case #6: *Conducting cylinder in uniform electric field.*

**Harmonic functions:**

Solutions of the Laplace equation are called harmonic functions. They are known to satisfy the mean-value theorem. A somewhat restricted version is stated in what follows without proof.

- ▷ Consider a region in space where the scalar field $\Phi(\mathbf{x})$ is a solution of $\nabla^2\Phi = 0$. Then the value $\Phi(\mathbf{x})$ is equal to the average value of $\Phi(\mathbf{x}')$ for points on a sphere $S$ of radius $R$ centered at $\mathbf{x}$:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi R^2} \oint_S da' \Phi(\mathbf{x}'),$$

  where $da'$ is the (scalar) element of area at position $\mathbf{x}'$ on the sphere.

- ▷ In situations with planar symmetry, the general solution of $d^2\Phi/dx^2 = 0$ is a linear function of the sole relevant coordinate: $\Phi(x) = ax + b$. The sphere reduces to two points. The value of $\Phi$ at the center of these two points is quite naturally the mean of the values at the endpoints:

$$\Phi(x) = \frac{1}{2}\Big[\Phi(x+R) + \Phi(x-R)\Big].$$

- ▷ In situations with translational in one direction, the solutions of the Laplace equation are functions of two relevant coordinates: $\Phi(\mathbf{x}) = \Phi(x, y)$. The sphere reduces to a circle $C$ and we can write,

$$\Phi(\mathbf{x}) = \frac{1}{2\pi R} \oint_C dl' \Phi(\mathbf{x}'),$$

  where $dl'$ is the (scalar) line segment at position $\mathbf{x}'$ on the circle.

- ▷ The real and imaginary parts of an analytic function $f(z)$ are harmonic functions and thus satisfy the Laplace equation separately. The mean-value theorem follows directly from the Cauchy integral:

$$f(a) = \frac{1}{2\pi\imath} \oint_C dz \frac{f(z)}{z-a} \quad \longrightarrow \quad f(a) = \frac{1}{2\pi} \int_0^{2\pi} d\phi f(a + re^{\imath\phi}).$$

The mean-value theorem implies that all minima of the electrostatic potential $\Phi(\mathbf{x})$ must be at the boundaries of the region within which it satisfies the Laplace equation. [lex215][lex216]

The mean-value theorem proves Earnshaw's theorem, which states the impossibility of stabilizing a charge configuration by electrostatic forces alone.