Electrostatics I [lln5]

Coulomb force between point charges:

Consider point charges $q_1, q_2, \ldots$ fixed to positions $\mathbf{x}_1, \mathbf{x}_2, \ldots$ Point charges are idealizations of charged particles.

Coulomb force: $\mathbf{F}_1 = \dfrac{q_1 q_2}{4\pi\epsilon_0} \dfrac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$ (exerted on q_1 by q_2).

Coulomb's law expresses the interaction force between two point charges. Interaction forces come in action-reaction pairs: $\mathbf{F}_2 = -\mathbf{F}_1$.

Permittivity constant: $\epsilon_0 = 8.854 \times 10^{-12}\mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}$.

Superposition principle: Interaction forces between point charges at rest are (vectorially) additive.

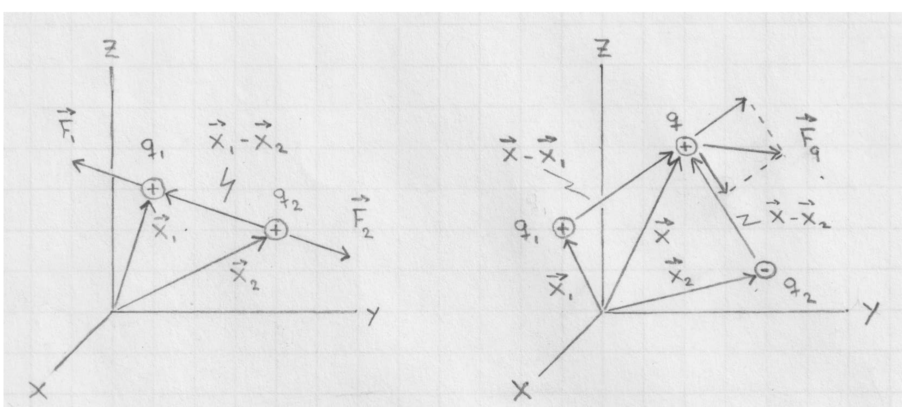
Force on point charge q at position $\mathbf{x}$ exerted by charges q_k at positions $\mathbf{x}_k$:

$$\mathbf{F}_q = \sum_k \frac{q q_k}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}, \tag{1}$$

Alternative notation using unit vectors and magnitudes:

$$\mathbf{r}_k \doteq \mathbf{x} - \mathbf{x}_k, \quad r_k \doteq |\mathbf{r}_k|, \quad \hat{\mathbf{r}}_k \doteq \frac{\mathbf{r}_k}{r_k}, \quad \mathbf{F}_q = \sum_k \frac{q q_k}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}_k}{r_k^2}$$

[lex187][lex218]



Instantaneous forces over distance are problematic. An alternative description of the Coulomb force facilitates the generalization to situations with moving charges.

Coulomb force mediated by electric field:

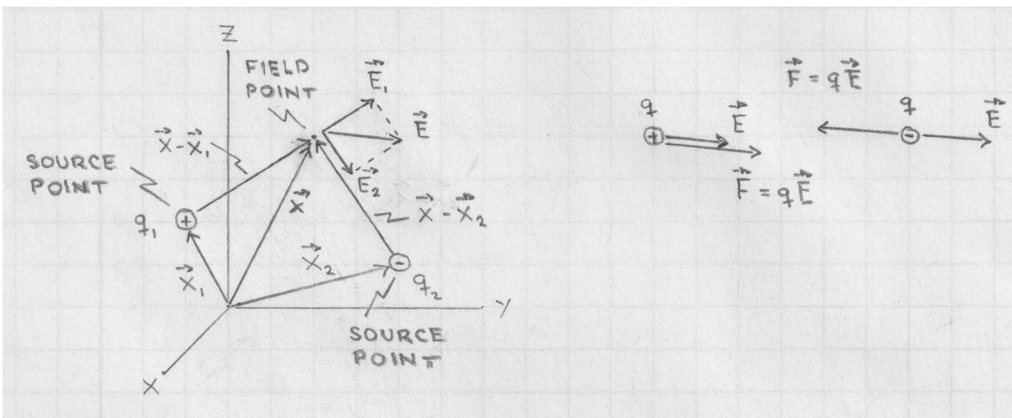
Point charges q_k fixed to positions $\mathbf{x}_k$ (source points) generate a static (time-independent) electric field $\mathbf{E}(\mathbf{x})$ at arbitrary field points $\mathbf{x}$:

$$\mathbf{E}(\mathbf{x}) = \sum_k \frac{q_k}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}. \tag{2}$$

A point charge q placed at position $\mathbf{x}$ experiences a force exerted by the local electric field:

$$\mathbf{F} = q\mathbf{E}(\mathbf{x}). \tag{3}$$

[lex39][lex128]



The force law (3) holds generally, even when charges are in motion, but an additional (magnetic) force comes into play.

Expression (2) for the electric forces can be generalized to situation with moving charges in the form of Gauss's law for the electric field.

Expression (1) for Coulomb interaction forces is restricted to electrostatics.

The notion that electric fields are generated by electric charges is only useful in electrostatics. Electrodynamics presents ways of generating an electric field which do not involve electric charges.

Electric charges are said to be a source of electric fields, which must be understood with this proviso. Electric fields interact with electric charges present in a region of space.

Charge densities:

Electric charge in ordinary matter is carried by electrons and protons in (discrete, negative or positive) units of the elementary charge,

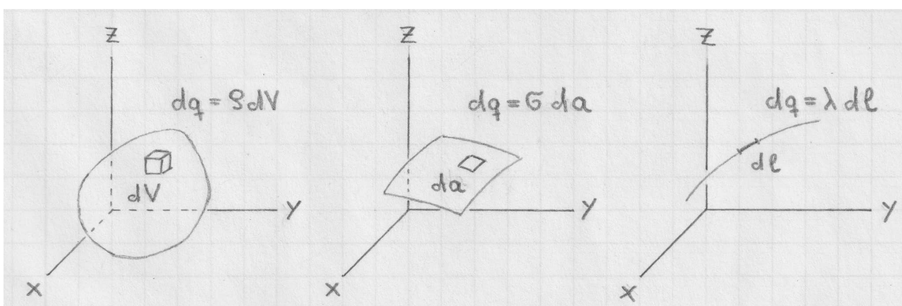
$$e = 1.602 \times 10^{-19}\text{C}.$$

For many purposes, it is justified and quite accurate to represent electric charge by a volume charge density $\rho(\mathbf{x})$, which averages the distribution of (discrete) electric charge over distances significantly larger than atomic radii.

Continuum limit of electric-field expression (2):

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{4}$$

If electric charge is distributed over a region in space which is effectively two-dimensional (e.g. a plane sheet or a thin spherical shell) or one-dimensional (e.g. a rod or a ring), such distributions are specified by a surface charge density $\sigma(\mathbf{x}')$ or a line charge density $\lambda(\mathbf{x}')$, respectively. [lex1]-[lex8]



In many applications, it is easy to adapt expression (4) to a two-dimensional integral across a surface S involving $\sigma(\mathbf{x}')$ or a one-dimensional integral along a curve C involving $\lambda(\mathbf{x}')$.

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S da'\,\sigma(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \quad \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_C dl'\,\lambda(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}.$$

Electric potential:

Mathematical identity: $\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$.

[gmd1-A] Application to electrostatic field (4):

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla\underbrace{\frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}}_{\Phi(\mathbf{x})} = -\nabla\Phi(\mathbf{x}).$$

Electric potential:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}. \tag{5}$$

[lex42][lex192]

Expression for discrete charges:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\sum_k \frac{q_k}{|\mathbf{x}-\mathbf{x}_k|}.$$

Mathematically, $\mathbf{E}(\mathbf{x})$ is a vector field and $\Phi(\mathbf{x})$ a scalar field. In general, scalar fields are more user-friendly than vector fields.

[gmd1-A] Electrostatic field is irrotational: $\nabla\times\mathbf{E}(\mathbf{x}) = \nabla\times\big[-\nabla\Phi(\mathbf{x})\big] = 0$.

Relation between electrostatic field and electric potential: [lex40]

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}), \quad \Phi(\mathbf{x}) = \Phi(\mathbf{x}_0) - \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x}'\cdot\mathbf{E}(\mathbf{x}'). \tag{6}$$

The attribute $\nabla\times\mathbf{E} = 0$ guarantees that the integral is path-independent –
[gmd1-B] a consequence of Stokes' theorem.

Gauss's law for the electric field:

Mathematical identity: $\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.

Application to electric potential (5) yields ...

Poisson equation: $-\nabla^2\Phi(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}$ $\quad [\nabla^2\Phi = \nabla\cdot(\nabla\Phi) = -\nabla\cdot\mathbf{E}]$.

Gauss's law: $\nabla\cdot\mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}$ (differential version).

Gauss's law is more general than Coulomb's law. Gauss's law remains valid when charges are in motion, i.e. in electrodynamics.

Electric flux:

Consider a surface S of arbitrary shape divided into infinitesimal elements of area da, represented as vectors $d\mathbf{a}$ directed perpendicular to the surface.

For open surfaces, one of two directions is chosen. For closed surfaces the convention is that $d\mathbf{a}$ points toward the outside.

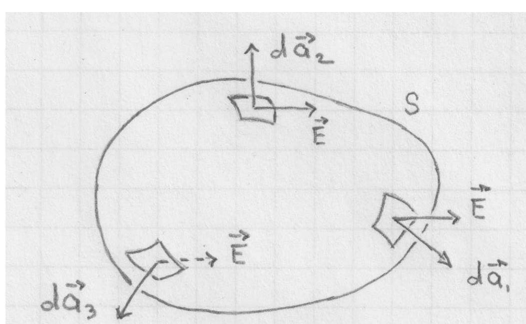
Electric flux through an arbitrary surface S:[1] $\Phi_E \doteq \int_S d\mathbf{a}\cdot\mathbf{E}$.

Application of Gauss's theorem and the differential version of Gauss's law:

$$\oint_S d\mathbf{a}\cdot\mathbf{E} = \int_V d^3x\, \underbrace{\nabla\cdot\mathbf{E}(\mathbf{x})}_{\rho(\mathbf{x})/\epsilon_0} = \frac{1}{\epsilon_0}\underbrace{\int_V d^3x\,\rho(\mathbf{x})}_{Q_{\rm in}}.$$

Gauss's law: $\displaystyle\oint_S d\mathbf{a}\cdot\mathbf{E} = \frac{Q_{\rm in}}{\epsilon_0}$ (integral version).

The electric flux through a closed surface of any shape is related to the net
[lex41] charge $Q_{\rm in}$ inside, irrespective of motion and the presence of charges outside.



The flux contribution from an element of area da is positive, zero, or negative, depending on whether the angle between the vectors $\mathbf{E}$ and $d\mathbf{a}$ is smaller than, equal to, or larger than 90°, respectively. [lex37][lex38]

[1]Note that the same Greek symbol is being used for electric potential $\Phi(\mathbf{x})$, for electric flux Φ_E, and (later) for magnetic flux Φ_B.

Electrostatic field at surfaces and interfaces:

How does the electrostatic field $\mathbf{E}$ change across the surface of a material or, more generally, across the interface between two materials? The answers for the tangential and normal components are obtained via different reasoning.

▷ *The tangential part $\mathbf{E}_\parallel$ of the electric field is continuous across the interface.*

This follows from the electrostatic condition, $\nabla \times \mathbf{E} = 0$, and Stokes' theorem applied to a flat rectangular loop with the long sides parallel to the interface on opposite sides and surrounding the open surface S_o.

[gmd1-B]

$$\oint_C d\mathbf{l} \cdot \mathbf{E} = \int_{S_\mathrm{o}} d\mathbf{a} \cdot \underbrace{\nabla \times \mathbf{E}}_{0} = 0 \quad \stackrel{dz\to 0}{\Rightarrow} \quad \Delta \mathbf{E}_\parallel = 0.$$

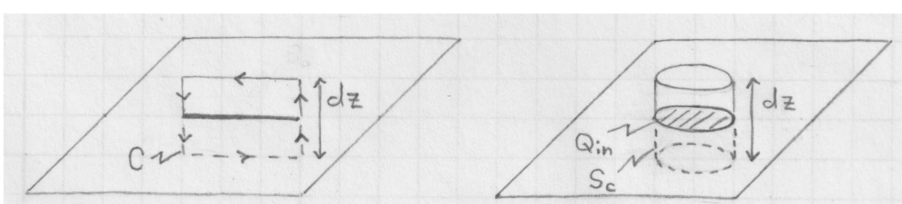
▷ *The normal part $\mathbf{E}_\perp$ of the electric field has a discontinuity in the amount of $|\Delta \mathbf{E}_\perp| = |\sigma|/\epsilon_0$ across the interface, where σ is the (local) surface charge density on the interface.*

This follows from Gauss's theorem and Gauss's law applied to a short pill box positioned across the interface with the two flat surfaces parallel to the interface and cutting out a patch S_p of the interface.

$$\oint_{S_\mathrm{c}} d\mathbf{a} \cdot \mathbf{E} = \int_V d^3x\, \nabla \cdot \mathbf{E} = \int_V d^3x\, \frac{\rho(\mathbf{x})}{\epsilon_0} = \frac{1}{\epsilon_0} \underbrace{\int_{S_\mathrm{p}} d^2x \sigma(\mathbf{x})}_{Q_\mathrm{in}}$$

$$\stackrel{dz\to 0}{\Rightarrow} \int_{S_\mathrm{p}} d^2x |\Delta E_z| = \frac{1}{\epsilon_0} \int_{S_\mathrm{p}} d^2x \sigma(\mathbf{x}) \quad \Rightarrow \quad |\Delta \mathbf{E}_\perp| = \frac{|\sigma|}{\epsilon_0}.$$

If $\sigma > 0$ ($\sigma < 0$) then the larger $\mathbf{E}_\perp$ is pointing away from (toward) the interface.



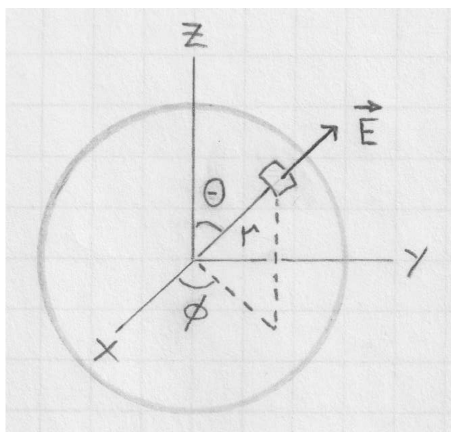
Electrostatic field determined via Gauss's law:

Gauss's law (integral version) can be used for the calculation of the electrostatic field of charged objects if specific symmetry conditions are satisfied.

▷ *Spherical symmetry*: Spherical coordinates r, θ, ϕ are in use. A charge distribution with full spherical symmetry, $\rho(r)$, generates an electric field in radial direction, $\mathbf{E} = E(r)\hat{\mathbf{r}}$. We use Gaussian surfaces of spherical shape.

The electric flux, Φ_E, is the product of the (unknown) electric field at radius r and the area, $4\pi r^2$, of the Gaussian sphere. Gauss's law relates the electric flux through the Gaussian sphere to the (known) net charge inside: [lex46]

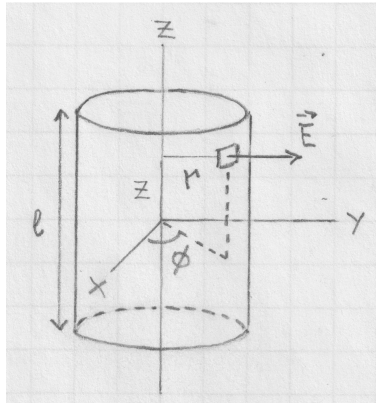
$$\Phi_E = 4\pi r^2 E(r) = \frac{1}{\epsilon_0}\int_0^r dr'(4\pi r'^2)\rho(r') = \frac{Q_{\rm in}}{\epsilon_0}.$$



▷ *Cylindrical symmetry*: Cylindrical coordinates r, ϕ, z are in use. A charge distribution with azimuthal and translational symmetry, $\rho(r)$, generates an electric field in radial direction, $\mathbf{E} = E(r)\hat{\mathbf{r}}$. We use Gaussian surfaces of cylindrical shape.

The electric flux, Φ_E, is the product of the (unknown) electric field at radius r and the area, $2\pi rl$, of the curved part of the Gaussian cylinder. The two flat portions of the Gaussian cylinder do not contribute electric flux of the radial field. Gauss's law relates the electric flux through the Gaussian cylinder to the (known) net charge inside: [lex47]

$$\Phi_E = 2\pi rlE(r) = \frac{1}{\epsilon_0}\int_0^r dr'(2\pi r'l)\rho(r') = \frac{Q_{\rm in}}{\epsilon_0}.$$

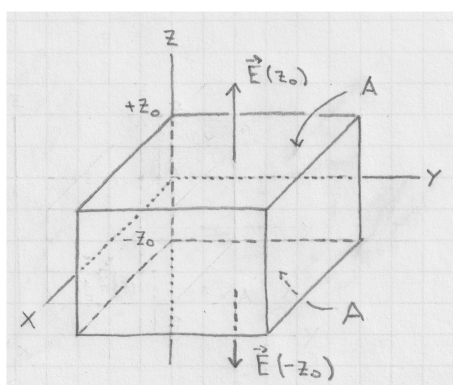


▷ *Planar symmetry*: Cartesian coordinates x, y, z are in use. A charge distribution with planar symmetry, $\rho(z)$ at $|z| \leq z_0$, generates an electric field in z-direction. We use a Gaussian surface in the shape of a box placed as shown.

The electric flux, Φ_E, is the sum of the products of the (unknown) field $\mathbf{E}$ at positions $\pm z_0$ with the area vectors $\mathbf{A}$ of the top or bottom sides of the box. The sides of the box perpendicular to the xy-plane do not contribute electric flux of the vertical field. Gauss's law relates the electric flux through the Gaussian box to the net charge inside: [lex189]

$$\Phi_E = A\big[E_z(z_0) - E_z(-z_0)\big] = 2AE_z(z_0) = \frac{A}{\epsilon_0}\int_{-z_0}^{+z_0} dz'\rho(z') = \frac{Q_{\text{in}}}{\epsilon_0}.$$

The relation $E_z(-z_0) = -E_z(z_0)$ holds for arbitrary $\rho(z')$ at $|z'| \leq z_0$.



Alternatively, we can take advantage of the fact that uniformly charged plane sheets generate uniform electric fields on either side:

$$E_z(z) = \frac{1}{2\epsilon_0}\int_{-z_0}^{z_0} dz'\rho(z')\text{sgn}(z - z').$$

Electrostatic energy:

If a region of space contains an electrostatic field $\mathbf{E}(\mathbf{x})$, it also contains an electric potential $\Phi(\mathbf{x})$. The two attributes of space are related to each other via (6).

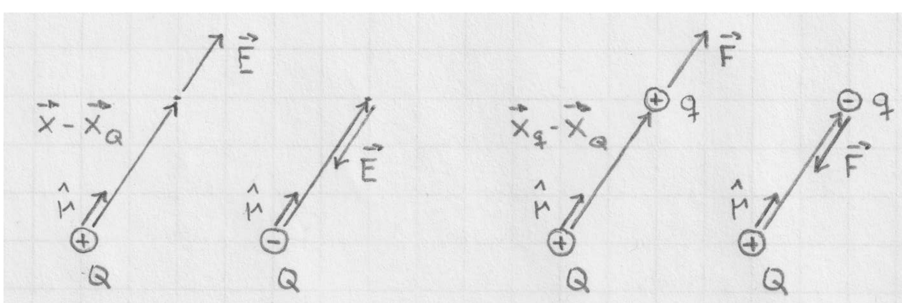
A particle with charge q placed at position $\mathbf{x}$ experiences the force $\mathbf{F} = q\mathbf{E}(\mathbf{x})$ and has the potential energy $U = q\Phi(\mathbf{x})$.

In electrostatics, the source of the electric field $\mathbf{E}(\mathbf{x})$ and the electric potential $\Phi(\mathbf{x})$ are other electric charges already positioned in space.

The table illustrates a particular case: one particle with charge Q is the source and another particle with charge q experiences field and potential generated by that source.

	attribute of		SI unit
field	space	$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\lvert\mathbf{x}-\mathbf{x}_Q\rvert^2}\,\hat{\mathbf{r}}$	[N/C]=[V/m]
potential	space	$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\lvert\mathbf{x}-\mathbf{x}_Q\rvert}$	[V]=[J/C]
force	particle	$\mathbf{F} = q\mathbf{E}(\mathbf{x}_q) = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\lvert\mathbf{x}_q-\mathbf{x}_Q\rvert^2}\,\hat{\mathbf{r}}$	[N]
energy	particle	$U = q\Phi(\mathbf{x}_q) = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\lvert\mathbf{x}_q-\mathbf{x}_Q\rvert}$	[J]

The unit vector $\hat{\mathbf{r}}$ always points away from the source.



The roles of the two particles are interchangeable. The force $\mathbf{F}$ represents an action-reaction pair. The energy U can be interpreted as the interaction potential energy of two charged particles.

Potential interaction energy of a discrete charge distribution:

$$U_{\text{int}} = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|}. \tag{7}$$

If we successively place three charges q_1, q_2, q_3 into a region of space, the first charge experiences no potential, the second charge experiences the potential of the first charge, and the third charge experiences the potential of the first two charges. The total interaction potential energy thus calculated,

$$U_{\text{int}} = 0 + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_3}{|\mathbf{x}_1 - \mathbf{x}_3|} + \frac{q_2 q_3}{|\mathbf{x}_2 - \mathbf{x}_3|}\right],$$

is equivalent to the general expression (7).

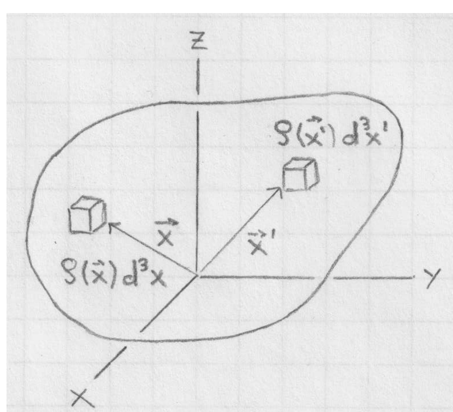
Potential energy of a continuous charge distribution $\rho(\mathbf{x})$:

$$U = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int d^3x \int d^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{2}\int d^3x\, \rho(\mathbf{x})\Phi(\mathbf{x}), \tag{8}$$

where the electric potential (5),

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

is generated by the same charge distribution $\rho(\mathbf{x})$.



Expression (7) and (8) are not equivalent. The latter includes, in addition to the interaction potential energy, also the potential self-energy. [lex120]

The inequivalence becomes more transparent after a transformation of the integrand in expression (8):

$$\rho(\mathbf{x})\Phi(\mathbf{x}) \overset{(a)}{=} -\epsilon_0\big[\nabla^2\Phi(\mathbf{x})\big]\Phi(\mathbf{x}) \overset{(b)}{=} \epsilon_0 \underbrace{\big[\nabla\Phi(\mathbf{x})\big]^2}_{(c)} - \epsilon_0\nabla\cdot\underbrace{\big[\Phi(\mathbf{x})\nabla\Phi(\mathbf{x})\big]}_{(d)}.$$

(a) Use the Poisson equation, $-\nabla^2\Phi(\mathbf{x}) = \rho(\mathbf{x})/\epsilon_0$.

[gmd1-B] (b) Use the mathematical identity, $\nabla\cdot(g\mathbf{F}) = g\nabla\cdot\mathbf{F} + \nabla g\cdot\mathbf{F}$.
Set $g \doteq \Phi$ and $\mathbf{F} \doteq \nabla\Phi \quad \Rightarrow \quad \nabla\cdot(\Phi\nabla\Phi) = \Phi\nabla^2\Phi + (\nabla\Phi)^2$.

(c) Use $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$.

(d) This term vanishes upon integration by virtue of Gauss's theorem if the surface is moved out to infinity, where the potential vanishes:

$$\int d^3x\nabla\cdot\big[\Phi(\mathbf{x})\nabla\Phi(\mathbf{x})\big] = \oint_S d\mathbf{a}\cdot\Phi(\mathbf{x})\nabla\Phi(\mathbf{x}) \rightsquigarrow 0,$$

Electrostatic potential energy:

$$\Rightarrow \; U = \frac{1}{2}\int d^3x\rho(\mathbf{x})\Phi(\mathbf{x}) = \frac{1}{2}\epsilon_0\int d^3x|\mathbf{E}(\mathbf{x})|^2 \geq 0. \tag{9}$$

The interaction potential energy U_{int} can be positive or negative. The total electrostatic energy U cannot be negative. It contains interaction energy and self-energy.

The electrostatic self-energy of point charges is infinite. This has been a lingering challenge in classical and quantum field theory.

Differential relations versus integral relations:

Local quantities:

$$\rho(\mathbf{x}) \quad [\mathrm{C/m^3}], \qquad \mathbf{E}(\mathbf{x}) \quad [\mathrm{V/m}], \qquad \Phi(\mathbf{x}) \quad [\mathrm{V}].$$

Differential relations:

$$\rho(\mathbf{x}) = \epsilon_0\nabla\cdot\mathbf{E}(\mathbf{x}), \qquad \rho(\mathbf{x}) = -\epsilon_0\nabla^2\Phi(\mathbf{x}), \qquad \mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}).$$

Integral relations:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}, \quad \Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|},$$

$$\Phi(\mathbf{x}) = \Phi(\mathbf{x}_0) - \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x}'\cdot\mathbf{E}(\mathbf{x}').$$

Electrostatic force on extended charged objects:

Consider a region of electrostatic field $\mathbf{E}_{\mathrm{ext}}(\mathbf{x})$ generated by unspecified sources.

Force on charged particle in this electric field: $\mathbf{F} = q\mathbf{E}_{\mathrm{ext}}(\mathbf{x})$. Does the electric field generated by the particle itself not contribute?

Consider instead an extended charged object with charge density $\rho(\mathbf{x})$.

Total electric field: $\mathbf{E}(\mathbf{x}) = \mathbf{E}_{\mathrm{ext}}(\mathbf{x}) + \mathbf{E}_{\mathrm{self}}(\mathbf{x})$.

Force exerted by external field: $\mathbf{F}_{\mathrm{ext}} = \displaystyle\int d^3x \rho(\mathbf{x})\mathbf{E}_{\mathrm{ext}}(\mathbf{x})$.

Electric field generated by the charged object:

$$\mathbf{E}_{\mathrm{self}}(\mathbf{x}) = -\nabla\Phi_{\mathrm{self}} = -\nabla\frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{4\pi\epsilon_0}\int d^3x' \rho(\mathbf{x}')\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}.$$

Force exerted by the field generated by the charged object itself:

$$\mathbf{F}_{\mathrm{self}} = \int d^3x \rho(\mathbf{x})\mathbf{E}_{\mathrm{self}}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x \int d^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}(\mathbf{x}-\mathbf{x}') = 0.$$

Note the antisymmetry of the integrand. The same conclusion holds for the charged particle.

Interaction energy and self-energy for extended charged objects:

Consider a continuous charge distribution in two parts, which may be spatially separated or overlapping:

$$\rho(\mathbf{x}) = \rho_1(\mathbf{x}) + \rho_2(\mathbf{x}).$$

The solution of the (linear) Poisson equation yields the electric potential in two parts and its (linear) gradient the electric field in two parts:

$$-\nabla^2\Phi = \frac{\rho}{\epsilon_0} \quad \Rightarrow \Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x}) \quad \Rightarrow -\nabla\Phi = \mathbf{E}(\mathbf{x}) = \mathbf{E}_1(\mathbf{x}) + \mathbf{E}_2(\mathbf{x}).$$

The potential energy associated with this charge configuration has three parts: two self-energies and one interaction energy:

$$\begin{aligned} U &= \frac{1}{2}\epsilon\int d^3x |\mathbf{E}_1(\mathbf{x}) + \mathbf{E}_2(\mathbf{x})|^2 \\ &= \underbrace{\frac{\epsilon_0}{2}\int d^3x |\mathbf{E}_1(\mathbf{x})|^2}_{U_{\mathrm{self}}^{(1)}} + \underbrace{\frac{\epsilon_0}{2}\int d^3x |\mathbf{E}_2(\mathbf{x})|^2}_{U_{\mathrm{self}}^{(2)}} + \underbrace{\epsilon_0\int d^3x |\mathbf{E}_1(\mathbf{x})\cdot\mathbf{E}_2(\mathbf{x})|}_{U_{\mathrm{int}}}. \end{aligned}$$

The transformation of the three terms employs steps used earlier:

$$U_{\text{self}}^{(1)} = \frac{1}{2}\int d^3x\rho_1(\mathbf{x})\Phi_1(\mathbf{x}), \quad U_{\text{self}}^{(2)} = \frac{1}{2}\int d^3x\rho_2(\mathbf{x})\Phi_2(\mathbf{x}),$$

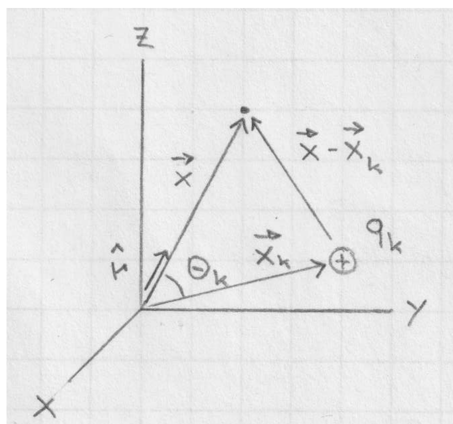
$$U_{\text{int}} = \int d^3x\rho_1(\mathbf{x})\Phi_2(\mathbf{x}) = \int d^3x\rho_2(\mathbf{x})\Phi_1(\mathbf{x}).$$

Multipole expansion:

Electric potential of two point charges: $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\sum_{k=1,2}\frac{q_k}{|\mathbf{x}-\mathbf{x}_k|}$.

Triangle spanned by the two vectors $\mathbf{x}$ and $\mathbf{x}_k$ at angle θ_k has sides,

$$r \doteq |\mathbf{x}|, \quad r_k \doteq |\mathbf{x}_k|, \quad |\mathbf{x}-\mathbf{x}_k| = \sqrt{r^2 - 2rr_k\cos\theta_k + r_k^2}.$$



$$\Rightarrow \quad \frac{1}{|\mathbf{x}-\mathbf{x}_k|} = \frac{1}{\sqrt{r^2 - 2r_kr\cos\theta_k + r_k^2}} = \frac{1}{r}\frac{1}{\sqrt{1+\kappa}}, \qquad \kappa \doteq \frac{r_k^2 - 2r_kr\cos\theta_k}{r^2}.$$

Binomial expansion: $\dfrac{1}{\sqrt{1+\kappa}} = 1 - \dfrac{1}{2}\kappa + \dfrac{3}{8}\kappa^2 + \ldots$

Unit vector: $\hat{\mathbf{r}} \doteq \dfrac{\mathbf{x}}{r} \quad \Rightarrow \hat{\mathbf{r}}\cdot\mathbf{x}_k = r_k\cos\theta_k$.

$$\Rightarrow \quad \frac{1}{r}\frac{1}{\sqrt{1+\kappa}} = \frac{1}{r}\left[1 - \frac{1}{2}\frac{r_k^2 - 2rr_k\cos\theta_k}{r^2} + \frac{3}{8}\frac{(r_k^2 - 2rr_k\cos\theta_k)^2}{r^4} + \cdots\right]$$
$$= \frac{1}{r} + \frac{r_k\cos\theta_k}{r^2} - \frac{1}{2}\frac{r_k^2}{r^3} + \frac{3}{2}\frac{(r_k\cos\theta_k)^2}{r^3} + \cdots$$

$$\Rightarrow \quad \frac{1}{|\mathbf{x}-\mathbf{x}_k|} = \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{x}_k}{r^2} + \frac{3(\hat{\mathbf{r}}\cdot\mathbf{x}_k)^2 - r_k^2}{2r^3} + \mathrm{O}\big(r^{-4}\big).$$

Electric potential of two point charges:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{|\mathbf{x}-\mathbf{x}_1|} + \frac{q_2}{|\mathbf{x}-\mathbf{x}_2|}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} + \frac{\hat{\mathbf{r}}\cdot\mathcal{Q}\cdot\hat{\mathbf{r}}}{r^3} + \ldots\right].$$

▷ Monopole: $Q = q_1 + q_2$ (scalar).

▷ Dipole: $\mathbf{p} = q_1\mathbf{x}_1 + q_2\mathbf{x}_2$ (vector).

▷ Quadrupole: $\mathcal{Q} = \frac{q_1}{2}\big[3\mathbf{x}_1\mathbf{x}_1 - r_1^2\mathcal{I}\big] + \frac{q_2}{2}\big[3\mathbf{x}_2\mathbf{x}_2 - r_2^2\mathcal{I}\big]$ (tensor).

Mathematically, the multipole expansion of the electric potential involves multipole moments $\mathcal{Q}_n$ in the form of tensors of rank 0 ($\mathcal{Q}_0 = Q$, scalar), rank 1 ($\mathcal{Q}_1 = \mathbf{p}$, vector), rank 2 ($\mathcal{Q}_2 = \mathcal{Q}$), and higher.

The dyadic product $\mathbf{xx}$ is a tensor of rank 2, as is the unit tensor $\mathcal{I}$:

$$\mathbf{xx} = \begin{pmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{pmatrix}, \qquad \mathcal{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Tensors of rank n have n indices.

Dot products between tensors are contractions, resulting in a reduction of tensor rank. All terms in the expansion of $\Phi(\mathbf{x})$ are scalars. In the second term we have a contraction of the rank-1 tensor $\mathbf{p}$ and in the third term two contractions of the rank-2 tensor $\mathcal{Q}$.

Electric multipole moments of discrete and continuous charge distributions:

▷ Monopole: $Q = \sum_k q_k, \quad Q = \int d^3x\,\rho(\mathbf{x}).$

▷ Dipole: $\mathbf{p} = \sum_k q_k\mathbf{x}_k, \quad \mathbf{p} = \int d^3x\,\rho(\mathbf{x})\,\mathbf{x}.$ [lex9][lex188]

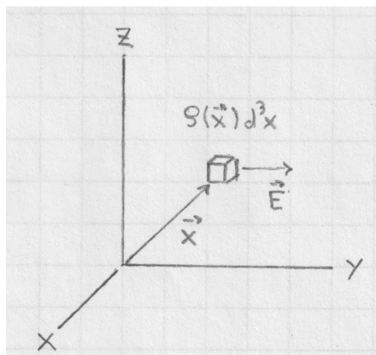
▷ Quadrupole: $\mathcal{Q} = \sum_k \frac{q_k}{2}\big(3\mathbf{x}_k\mathbf{x}_k - r_k^2\mathcal{I}\big), \quad \mathcal{Q} = \int d^3x\,\rho(\mathbf{x})\frac{1}{2}\big(3\mathbf{xx} - r^2\mathcal{I}\big).$

Torque and force on electric dipole:

Torque $\mathbf{N}$ acting on electric dipole $\mathbf{p}$ in electric field $\mathbf{E}$:

$$\mathbf{N} = \int d^3x\, \rho(\mathbf{x})\mathbf{x} \times \mathbf{E}(\mathbf{x}) \overset{(a)}{\rightsquigarrow} \mathbf{p} \times \mathbf{E},$$

- force acting on charge element: $d\mathbf{F} = \mathbf{E}(\mathbf{x})\rho(\mathbf{x})d^3x$,
- torque acting on charge element: $d\mathbf{N} = \mathbf{x} \times d\mathbf{F}$,
- assumption (a): $\mathbf{E}$ varies negligibly across volume occupied by dipole.



Potential energy $U(\mathbf{x})$ of electric dipole $\mathbf{p}$ in electric field $\mathbf{E}(\mathbf{x})$:

$$U = \int d^3x' \rho(\mathbf{x}')\Phi(\mathbf{x}') \overset{(b)}{=} \int d^3x' \rho(\mathbf{x}')\mathbf{x}' \cdot \nabla\Phi(\mathbf{x}) = -\mathbf{p} \cdot \mathbf{E}(\mathbf{x}),$$

- step (b) uses $\Phi(\mathbf{x}') \simeq \Phi(\mathbf{x}) + (\mathbf{x}' - \mathbf{x}) \cdot \nabla\Phi(\mathbf{x})$,
- step (b) also uses $\int d^3x' \rho(\mathbf{x}') = 0$ (zero net charge).

Force $\mathbf{F}$ acting on electric dipole $\mathbf{p}$ in electric field $\mathbf{E}$: [lex195][lex197]

$$\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x}) = \nabla[\mathbf{p} \cdot \mathbf{E}(\mathbf{x})] \overset{(c)}{=} (\mathbf{p} \cdot \nabla)\mathbf{E}(\mathbf{x}),$$

- step (c) uses $\nabla \times \mathbf{E}(\mathbf{x}) = 0$,
- step (c) uses $\mathbf{p} = \text{const}$ (not position-dependent),
- step (c) uses mathematical identity:

[gmd1-A] $$\nabla(\mathbf{p} \cdot \mathbf{E}) = \mathbf{p} \times \underbrace{(\nabla \times \mathbf{E})}_{0} + \mathbf{E} \times \underbrace{(\nabla \times \mathbf{p})}_{0} + (\mathbf{p} \cdot \nabla)\mathbf{E} + \underbrace{(\mathbf{E} \cdot \nabla)\mathbf{p}}_{0},$$