Quantum Optics III [lln27]

Resonant light-atom interactions:

Light induces electric dipole oscillations in atoms, which then radiate at the same frequency. Circumstances favorable to enhanced interactions due to matching attributes are named resonance.

Here we investigate what happens when an atom is irradiated with a light beam in resonance with a particular electronic level spacing.

The focus is on a two-level system: $E_2 > E_1, \quad E_2 - E_1 = \hbar\omega$.

The pair of resonant levels is considered in full detail. Other atomic levels are accounted for summarily as a source of damping.

Questions of interest:

- What goes on in an atom during the absorption process?
- What is spontaneous emission really? Is it a random process as assumed by Einstein's theory or is it stimulated by vacuum fluctuations?

Pure versus mixed quantum states:

Consider a physical ensemble of N noninteracting two levels systems: eigenstates $|1\rangle$, $|2\rangle$ with energies E_1, E_2, respectively.

- A member of the ensemble is said to be in a *pure state* if it is a coherent superposition of eigenstates: $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$. This includes the stationary states $|1\rangle$ (if $c_2 = 0$) and $|2\rangle$ (if $c_1 = 0$).
- The ensemble with all N members in a stationary state is said to be in a *mixed state* with, say, N_1 members in eigenstate $|1\rangle$ and N_2 members in eigenstate $|2\rangle$.

The difference between the two states is encoded in the density operator:

$$\rho_{\rm P} = \begin{pmatrix} c_1^* c_1 & c_1 c_2^* \\ c_1^* c_2 & c_2 c_2^* \end{pmatrix}, \quad \rho_{\rm M} = \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{pmatrix}.$$

If $|c_1|^2 = N_1/N$ and $|c_2|^2 = N_2/N$, the probability of measuring the energy E_1 or E_2 is the same for the pure state and the mixed state.

Stationary states have independent phases. The light links the phases between the two levels, causing transitions between them: absorption and stimulated emission.

Bloch sphere:

Coherent superposition states,

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle, \quad |c_1|^2 + |c_2|^2 = 1,$$

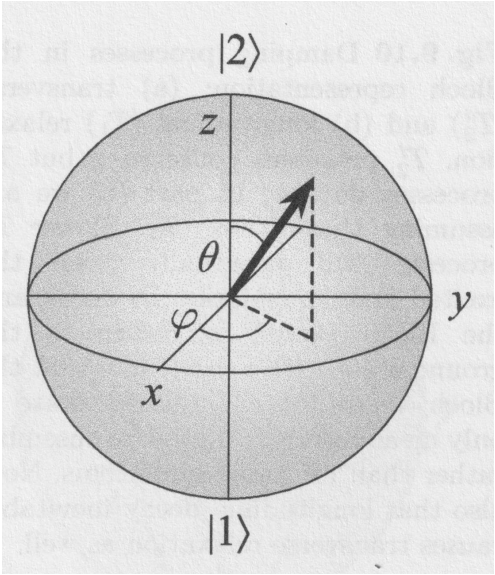
can be represented by points on a unit sphere, named *Bloch sphere*, with the stationary states $|1\rangle$ and $|2\rangle$ at the South Pole and North Pole, respectively.

$$x = \sin\theta\cos\varphi,$$
$$y = \sin\theta\sin\varphi,$$
$$z = \cos\theta.$$

$$x = 2\Re[c_1 c_2],$$
$$y = 2\Im[c_1 c_2],$$
$$z = |c_2|^2 - |c_1|^2.$$

$$\Updownarrow$$

$$c_1 = \sin\frac{\theta}{2}, \quad c_2 = \cos\frac{\theta}{2}\, e^{\imath\varphi}.$$



[image from Fox 2014]

The mixed ensemble state identified earlier would, by contrast, be represented by a pointed on the axis connecting the poles of the Bloch sphere.

Schrödinger equation of interacting system:

Atomic two level system subject to interaction potential:

$$\mathcal{H}\Psi = \imath\hbar\frac{\partial\Psi}{\partial t}, \quad \mathcal{H} = \mathcal{H}_0 + V(\mathbf{r}, t).$$

Nonstationary solution: $\Psi(\mathbf{r}, t) = c_1(t)\psi_1(\mathbf{r})e^{-\imath E_1 t/\hbar} + c_2(t)\psi_2(\mathbf{r})e^{-\imath E_2 t/\hbar}$.

Stationary states of $\mathcal{H}_0$: $\mathcal{H}_0\psi_i(\mathbf{r}) = E_i\psi_i(\mathbf{r}) \quad : \ i = 1, 2.$

Normalization: $\int d^3r\psi_i^*(\mathbf{r})\psi_j(\mathbf{r}) = \delta_{ij}$.

Resonance: $\omega_0 = (E_2 - E_1)/\hbar, \quad \omega = \omega_0 + \delta\omega, \quad |\delta\omega| \ll \omega_0.$

Substitute ansatz for nonstationary solution into Schrödinger equation and extract contribution due to interaction potential $V(\mathbf{r},t)$:

$$\begin{aligned}\big[\mathcal{H}_0+V\big]\big[c_1\psi_1 e^{-\imath E_1 t/\hbar}+c_2\psi_2 e^{-\imath E_2 t/\hbar}\big]&\\ =\imath\hbar\left\{\left[\dot{c}_1-\frac{\imath E_1 c_1}{\hbar}\right]e^{-\imath E_1 t/\hbar}+\left[\dot{c}_2-\frac{\imath E_2 c_2}{\hbar}\right]e^{-\imath E_2 t/\hbar}\right\}&\end{aligned}$$

$$\mathcal{H}_0\big[c_1\psi_1 e^{-\imath E_1 t/\hbar}+c_2\psi_2 e^{-\imath E_2 t/\hbar}\big]=c_1E_1\psi_1 e^{-\imath E_1 t/\hbar}+c_2E_2\psi_2 e^{-\imath E_2 t/\hbar}.$$

$$\Rightarrow\ V\big[c_1\psi_1 e^{-\imath E_1 t/\hbar}+c_2\psi_2 e^{-\imath E_2 t/\hbar}\big]=\imath\hbar\big\{\dot{c}_1\psi_1 e^{-\imath E_1 t/\hbar}+\dot{c}_2\psi_2 e^{-\imath E_2 t/\hbar}\big\}.$$

Use normalization of stationary states to infer coupled linear first-order ODEs for the coefficients of the nonstationary solution:

$$\begin{aligned}\dot{c}_1(t)&=-\frac{\imath}{\hbar}\big[c_1(t)V_{11}(t)+c_2(t)V_{12}(t)e^{-\imath\omega_0 t}\big],\\ \dot{c}_2(t)&=-\frac{\imath}{\hbar}\big[c_1(t)V_{21}(t)e^{\imath\omega_0 t}+c_2(t)V_{22}(t)\big],\end{aligned}$$

Interaction matrix elements: $V_{ij}(t)\doteq\int d^3r\,\psi_i^*(\mathbf{r})V(\mathbf{r},t)\psi_j(\mathbf{r})$.

Electric dipolar interaction (E1) dominates: $V(\mathbf{r},t)=e\mathbf{r}\cdot\mathbf{E}(t)$.

Linearly polarized light: $V(\mathbf{r},t)=exE_0\cos(\omega t)=\dfrac{exE_0}{2}\big[e^{\imath\omega t}+e^{-\imath\omega t}\big]$.

$$\Rightarrow\ V_{ij}(t)=\frac{eE_0}{2}\big[e^{\imath\omega t}+e^{-\imath\omega t}\big]\int d^3r\,\psi_i^*(\mathbf{r})x\psi_j(\mathbf{r})=-\frac{1}{2}E_0\mu_{ij}\big[e^{\imath\omega t}+e^{-\imath\omega t}\big].$$

Electric-dipole matrix elements: $\mu_{ij}=-e\int d^3r\,\psi_i^*(\mathbf{r})x\psi_j(\mathbf{r})$.

Odd parity of operator requires that $\mu_{11}=\mu_{22}=0,\ \ \mu_{12}=\mu_{21}\doteq\mu$.

A nonzero μ requires that atomic states 1 and 2 have opposite parity.

Coupled ODEs for the this particular interaction:

$$\begin{aligned}\dot{c}_1(t)&=\frac{\imath}{2}\Omega_{\mathrm{R}}\big[e^{\imath(\omega-\omega_0)t}+e^{-\imath(\omega+\omega_0)t}\big]c_2(t),\\ \dot{c}_2(t)&=\frac{\imath}{2}\Omega_{\mathrm{R}}\big[e^{-\imath(\omega-\omega_0)t}+e^{\imath(\omega+\omega_0)t}\big]c_1(t),\end{aligned}\qquad(1)$$

Rabi (angular) frequency: $\Omega_{\mathrm{R}}=\left|\dfrac{\mu E_0}{\hbar}\right|$.

Near resonance the first factor inside the square bracket is a slow oscillation and the second factor a fast oscillation.

Einstein coefficients in the weak-field limit:

The focus is on absorption processes under the assumption that the interaction does not significantly affect the population of the lower level.

Weak-field assumption: $\dot{c}_1(0) \simeq 0$.

Initial conditions: $c_1(0) = 1,\ c_2(0) = 0$.

Near-resonance condition: $\delta\omega \doteq \omega - \omega_0 \ll \omega_0$.

Reduction to single ODE: $\dot{c}_2(t) = \frac{\imath}{2}\Omega_{\mathrm{R}}\left[e^{-\imath(\omega-\omega_0)t} + e^{\imath(\omega+\omega_0)t}\right]$.

$$\Rightarrow\ c_2(t) = \frac{\imath}{2}\Omega_{\mathrm{R}}\left[\frac{e^{-\imath\delta\omega t}-1}{-\imath\delta\omega} + \frac{e^{\imath(\omega+\omega_0)t}-1}{\imath(\omega+\omega_0)}\right] \overset{\delta\omega\ll 1}{\longrightarrow} \frac{1}{2}\Omega_{\mathrm{R}}\frac{1-e^{-\imath\delta\omega t}}{\delta\omega}.$$

The last step, which neglects fast low-amplitude oscillations, is named *rotating wave approximation*.

Raw weak-field absorption rate: $|c_2(t)|^2 = \frac{\Omega_{\mathrm{R}}^2}{4}\frac{\sin^2(\delta\omega t/2)}{(\delta\omega/2)^2}, \quad \Omega_{\mathrm{R}} = \left|\frac{\mu E_0}{\hbar}\right|$.

A more realistic result averages over frequencies near resonance:[1]

$$\begin{aligned}|c_2(t)|^2 &= \frac{\mu^2 E_0^2}{4\hbar^2}\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} d\omega\, \frac{2u(\omega)}{\epsilon_0 E_0^2}\frac{\sin^2\big((\omega-\omega_0)t/2\big)}{(\omega-\omega_0)^2/4} \\ &\simeq \frac{\mu^2 u(\omega_0)}{2\epsilon_0\hbar^2}\underbrace{\int_{-\infty}^{+\infty} d\omega\left(\frac{\sin(\omega t/2)}{\omega/2}\right)^2}_{2\pi t} = \frac{\pi\mu^2}{\epsilon_0\hbar^2}u(\omega_0)t.\end{aligned}$$

The linear time-dependence is indicative of a steady rate of absorption.

Einstein absorption coefficient [lln24]: $\frac{dN_2}{dt} = -\frac{dN_1}{dt} = B_{12}u(\omega_0)N_2$.

Relation to transition rate:[2] $W_{12} = B_{12}u(\omega_0) = \frac{|c_2(t)|^2}{3t}$.

$$\Rightarrow\ B_{12} = \frac{\pi\mu^2}{3\epsilon_0\hbar^2}.$$

The coefficients A_{21} and B_{21} for spontaneous and stimulated emissions, respectively, can be inferred from B_{12} as described in [lln24].

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[1] The rescaled spectral energy density must satisfy $\int d\omega u(\omega) = \epsilon_0 E_0^2/2$.

[2] The factor $\frac{1}{3}$ is the result of directional averaging.

Rabi oscillations in the strong-field limit:

Strong-field situations can be realized with light from powerful laser beams. A simple solution of the coupled ODEs (1) can be found

- if we invoke the rotating-wave approximation (neglect fast oscillations),
- if the light drives the two-level system at resonance ($\omega = \omega_0$).

Simplified coupled ODEs: $\dot{c}_1(t) = \frac{\imath}{2}\,\Omega_\mathrm{R}\,c_2(t), \quad \dot{c}_2(t) = \frac{\imath}{2}\,\Omega_\mathrm{R}\,c_1(t).$

Separated ODEs: $\ddot{c}_1(t) = -\frac{1}{4}\Omega_\mathrm{R}^2\,c_1(t), \quad \ddot{c}_2(t) = -\frac{1}{4}\Omega_\mathrm{R}^2\,c_2(t).$

Solutions with $c_1(0) = 1$ and $c_2(0) = 0$:

$$c_1(t) = \cos\left(\frac{\Omega_\mathrm{R} t}{2}\right), \quad c_2(t) = \sin\left(\frac{\Omega_\mathrm{R} t}{2}\right), \quad \Omega_\mathrm{R} = \left|\frac{\mu E_0}{\hbar}\right|.$$

Occupation probabilities of the two levels are subject to *Rabi oscillations*, also named *Rabi flopping*:

$$|c_1(t)|^2 = \cos^2\left(\frac{\Omega_\mathrm{R} t}{2}\right), \quad |c_2(t)|^2 = \sin^2\left(\frac{\Omega_\mathrm{R} t}{2}\right).$$

Laser light that is slightly off resonance affects both the period and the amplitude of the Rabi oscillations [lex156]:

$$|c_2(t)|^2 = \left(\frac{\Omega_\mathrm{R}}{\Omega}\right)^2 \sin^2\left(\frac{\Omega t}{2}\right), \quad \Omega^2 = \Omega_\mathrm{R}^2 + (\delta\omega)^2.$$

The condition for persistent coherence, on which the observation of Rabi oscillations depends, can be expressed by the inequality,

$$\Omega_\mathrm{R} = \left|\frac{\mu E_0}{\hbar}\right| \gg \frac{2\pi}{\tau}, \quad (\tau : \text{radiative lifetime}).$$

The causes for the damping of Rabi oscillations split into two parts:

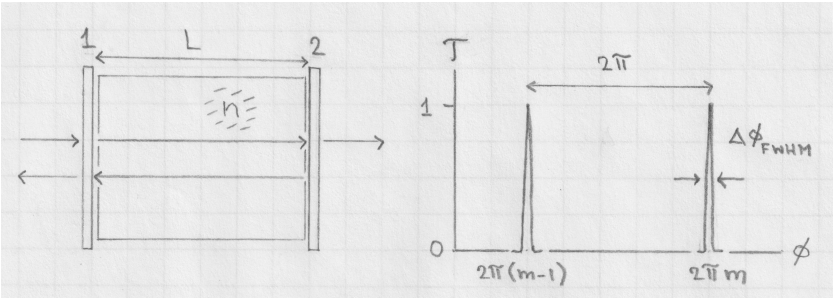
- Decay of level population (longitudinal relaxation), characterized by time constant T_1. Spontaneous emission is a major contribution. The radiative lifetime τ is an upper limit of T_1. Nonradiative processes cause a further reduction.
- Dephasing (transverse relaxation) is characterized by the time constant T_2. Dephasing is population conserving. Elastic collisions in gases or phonon scattering in solids are major causes.

Optical cavities:

The interaction between atoms and photons is enhanced if it takes place inside a resonant cavity.

A planar cavity has a dielectric material between two parallel mirrors:

- ▷ L: adjustable separation between mirrors 1 and 2,
- ▷ n: refractive index of dielectric material,
- ▷ R_1, R_2: reflectivities of mirrors,
- ▷ λ: wavelength of incident light (from left).



In high-resolution spectroscopy, this is a *Fabry-Perot* interferometer.

- Cavity transmission: $\mathcal{T} = \left[1 + \left(\dfrac{2\mathcal{F}}{\pi}\right)^2 \sin^2 \dfrac{\phi}{2}\right]^{-1}$.
- Round-trip phase shift: $\phi = \omega t = \omega \dfrac{2L}{c/n} = 2nkL = \dfrac{4\pi nL}{\lambda}$.
- Cavity finesse: $\mathcal{F} = \dfrac{\pi (R_1 R_2)^{1/4}}{1 - \sqrt{R_1 R_2}}$.
- Resonance condition: $\phi = 2\pi m \quad \Rightarrow \; L = \dfrac{m\lambda}{2n} \quad : \; m = 1, 2, \ldots$.
- Transmission at resonance: $\mathcal{T} = 1$.
- For high finesse ($\mathcal{F} \gg 1$), the transmission $\mathcal{T}$ is sharply peaked at points of resonance (see sketch).
- Full width at half maximum: $\left(\dfrac{2\mathcal{F}}{\pi}\right) \sin \dfrac{\phi}{2} = 1 \quad \Rightarrow \; \Delta\phi_{\text{FWHM}} = \dfrac{2\pi}{\mathcal{F}}$.
- Resolving power: $\mathcal{F} = \dfrac{2\pi}{\Delta\phi_{\text{FWHM}}} = \dfrac{\text{separation of peaks}}{\text{FWHM of peak}}$.
- Resonant frequencies: $\omega_m = \dfrac{2\pi c}{\lambda} = \dfrac{m\pi c}{nL}$.

Resonant light in the cavity is in phase. Waves interfere constructively, producing large amplitudes.

Spectral width of resonant modes: Use $\omega \propto \phi$ and $\omega_m - \omega_{m-1} = \pi c/nL$,

$$\frac{\Delta\omega}{\omega_m - \omega_{m-1}} = \frac{\Delta\phi_{\mathrm{FWHM}}}{2\pi} = \frac{1}{\mathcal{F}} \quad \Rightarrow \; \Delta\omega = \frac{\pi c}{n\mathcal{F}L}.$$

Cavity lifetime of photons: Use $R_1 = R_2 \doteq R$.

$$\frac{dN}{dt} = \frac{\Delta N}{\Delta t} = -\frac{N(1-R)}{nL/c} \quad \Rightarrow \; N(t) = N_0 e^{-t/\tau_{\mathrm{cav}}}, \quad \tau_{\mathrm{cav}} = \frac{nL}{c(1-R)}.$$

$N(1-R)$ is the loss during each mirror reflection and nL/c is the time between reflections.

Cavity photon loss rate at high-reflectivity ($R \simeq 1$):

$$\mathcal{F} \simeq \frac{\pi}{1-R}, \quad \kappa \doteq \frac{1}{\tau_{\mathrm{cav}}} = \frac{c(1-R)}{nL} \simeq \frac{\pi c}{n\mathcal{F}L} = \Delta\omega.$$

Quality factor: $Q \doteq \dfrac{\omega}{\Delta\omega} \simeq \dfrac{\omega}{\kappa}$.

The resonance condition is an attribute of the optical cavity alone, independent of external irradiation. In the absence of photons of a resonant mode, the cavity remains resonant for vacuum fluctuations of the same mode.

Two-level atom in optical cavity:

Consider a cavity tuned to resonate with a two-level atom at frequency ω. The interaction at resonance is then governed by three parameters with units s^{-1}, thus representing different time scales:

- cavity photon loss rate κ,
- non-resonant atomic decay rate γ,
- atom-photon coupling strength g_0.

It is useful to distinguish a weak-coupling regime, $g_0 \ll \max(\kappa, \gamma)$, and a strong-coupling regime, $g_0 \gg \max(\kappa, \gamma)$.

In the strong-coupling regime, photons emitted by atoms are likely to be re-absorbed before before they are lost from the cavity. In the weak-coupling regime, re-absorption is unlikely.

The photon loss rate κ is governed by the quality factor of the cavity: $\kappa = \omega/Q$. A high value of Q suppresses losses.

Contributions to the non-resonant decay rate γ include

- photons emitted at resonant frequency away from the direction of the resonant mode,
- photons emitted at different frequencies,
- non-radiative atomic transitions.

For the determination of the coupling strength g_0, we examine the interaction energy between the two-level atom and the vacuum electric field in the cavity:

- Electric dipole interaction: $\Delta U = |\mu_{12} E_{\text{vac}}|, \quad \mu_{12} = -e\langle 1|x|2\rangle$.
- Vacuum electric field:[3] $2\int d^3x\, \frac{1}{2}\epsilon_0 E_{\text{vac}}^2 = \frac{1}{2}\hbar\omega \quad \Rightarrow \; E_{\text{vac}} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V_0}}$.
- Measure for coupling strength: $\Delta U = \hbar g_0 \quad \Rightarrow \; g_0 = \sqrt{\frac{\mu_{12}^2 \omega}{2\epsilon_0 \hbar V_0}}$.
- For N atoms in a cavity, the strong-coupling condition requires that $\sqrt{N} g_0 \gg \max(\kappa, \gamma)$. This condition is rarely met unless N is large.

Weak-coupling regime:

We focus on spontaneous photon emission, first in free space, then in a resonating cavity. Free space is represented here by a large volume V_{L}.

- Fermi's golden rule for transition rate:[4] $W = \frac{2\pi}{\hbar^2}|M_{12}|^2 g(\omega)$.
- Density of photon modes [lln24]: $g(\omega) = \frac{\omega^2 V_{\text{L}}}{\pi^2 c^3}$.
- Matrix element for electric-dipole (E1) interaction:[5] $M_{12} = \langle \mathbf{p} \cdot \mathbf{E} \rangle$.

  $\Rightarrow \; |M_{12}|^2 = \frac{1}{3}\mu_{12}^2 E_{\text{vac}}^2 = \frac{\mu_{12}^2 \hbar\omega}{6\epsilon_0 V_{\text{L}}}; \quad E_{\text{vac}} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V_{\text{L}}}}, \quad \mu_{12} = -e\langle 1|x|2\rangle$.
- Einstein coefficient for spontaneous emission: $A_{21} = W = \frac{\mu_{12}^2 \omega^3}{3\pi\epsilon_0 \hbar c^3}$.

[3] E_{vac} thus defined represents the rms value.

[4] We have used $f(\epsilon)d\epsilon = g(\omega)d\omega$ for $\epsilon = \hbar\omega$, from which $f(\epsilon) = g(\omega)/\hbar$ follows.

[5] The factor $\frac{1}{3}$ represents orientational averaging.

Consistency with the previous result for the Einstein absorption coefficient, $B_{12} = \pi\mu_{12}^2/3\epsilon_0\hbar^2$, is confirmed if we use the relations (for $g_1 = g_2$) between the coefficients established in [lln24]:

$$A_{21} = \frac{\hbar\omega^3}{\pi^2 c^3} B_{21}, \quad B_{21} = B_{12}.$$

Purcell effect:

The *Purcell effect* describes the change in the rate of spontaneous emission from a two-level atom due to confinement in a single-model resonant cavity.

- Cavity resonance frequency: ω_c.
- Quality factor determines linewidth $\Delta\omega_\mathrm{c}$: $Q = \dfrac{\omega_\mathrm{c}}{\Delta\omega_\mathrm{c}}$.
- Lorentzian lineshape [lln24]: $g(\omega) = \dfrac{2}{\pi\Delta\omega_\mathrm{c}} \dfrac{(\Delta\omega_\mathrm{c})^2}{4(\omega-\omega_\mathrm{c})^2 + (\Delta\omega_\mathrm{c})^2}$.
- Transition frequency of two-level atom: $\omega_0 = (E_2 - E_1)/\hbar$.
- Resonance condition: $\omega_0 = \omega_\mathrm{c}, \quad \Rightarrow \; g(\omega_0) = \dfrac{2}{\pi\Delta\omega_\mathrm{c}} = \dfrac{2Q}{\pi\omega_0}$.
- Matrix element generalized: $|M_{12}|^2 = \xi^2\mu_{12}^2 E_\mathrm{vac}^2 = \xi^2 \dfrac{\mu_{12}^2\hbar\omega_\mathrm{c}}{2\epsilon_0 V_0}$.
- Cavity vacuum electric field: $E_\mathrm{vac} = \sqrt{\dfrac{\hbar\omega}{2\epsilon_0 V_0}}$.
- Orientational factor: $\xi \doteq \dfrac{|\mathbf{p}\cdot\mathbf{E}|}{|\mathbf{p}||\mathbf{E}|}$ with $\xi^2 \overset{\text{rand. orien.}}{\longrightarrow} \frac{1}{3}$.
- Cavity transition rate: $W_\mathrm{cav} = \dfrac{2Q\mu_{12}^2}{\hbar\epsilon_0 V_0}\xi^2 \underbrace{\dfrac{(\Delta\omega_\mathrm{c})^2}{4(\omega_0-\omega_\mathrm{c})^2 + (\Delta\omega_\mathrm{c})^2}}_{\to 1 \text{ at res.}}$.
- Purcell factor: $F_\mathrm{P} \doteq \dfrac{W_\mathrm{cav}}{W} = \dfrac{6\pi Q}{V_0}\dfrac{c^3}{\omega^3}\xi^2 \dfrac{(\Delta\omega_\mathrm{c})^2}{4(\omega_0-\omega_\mathrm{c})^2 + (\Delta\omega_\mathrm{c})^2}$.
- Change in wavelength: $\dfrac{c}{\omega} = \dfrac{1}{k} = \dfrac{\lambda_\mathrm{cav}}{2\pi}, \quad \lambda_\mathrm{cav} = \dfrac{\lambda}{n}$.
- Purcell factor at resonance: $F_\mathrm{P} = \dfrac{3Q}{4\pi^2 V_0}\left(\dfrac{\lambda}{n}\right)^3 \xi^2$.
- $F_\mathrm{P} > 1$ can be realized for high Q and small modal cavity volume V_0

Spontaneous photon emission is, in fact, emission stimulated by vacuum fluctuations, which can be enhanced in optical cavities at resonance.

Strong-coupling regime:

Optical cavities in the strong-coupling regime are the stuff of cavity QED, to be investigated more thoroughly in a later module.

The Jaynes-Cummings model is designed for the examination of the energy-level structure of a two-level atom interacting with a single quantized mode of radiation field.

- Hamiltonian: $\mathcal{H}_{\rm JC} = \hbar\omega_{\rm S} S_z + \hbar\omega_{\rm B} a^\dagger a + \Lambda(S_+ a + S_- a^\dagger)$.
- Energy scale for two-level atom: $\hbar\omega_{\rm S}$.
- Energy scale for radiation field: $\hbar\omega_{\rm B}$.
- Coupling strength: $\Lambda = \hbar g_0 = \hbar\sqrt{\dfrac{\mu_{12}^2\omega}{2\epsilon_0\hbar V_0}}$ (from earlier).
- Basis vectors: $|m,n\rangle, \quad : \ m = 0,1; \quad n = 0,1,2,\ldots$.
- Operators in action:

$$\begin{aligned}
&\left(\tfrac{1}{2} - S_z\right)|m,n\rangle = m|m,n\rangle, \quad S_+|m,n\rangle = \sqrt{m(1-m+1)}\,|m-1,n\rangle,\\
&S_-|m,n\rangle = \sqrt{(1-m)(m+1)}\,|m+1,n\rangle,\\
&a^\dagger|m,n\rangle = \sqrt{n+1}\,|m,n+1\rangle, \quad a|m,n\rangle = \sqrt{n}\,|m,n-1\rangle.
\end{aligned}$$

- Eigenstates for $\omega_{\rm S} = \omega_{\rm B} \doteq \omega$ [lex159]:

$$\begin{aligned}
|\psi_{1,0}\rangle &= |1,0\rangle,\\
|\psi_{1,n}\rangle &= \frac{1}{\sqrt{2}}\big[|1,n\rangle + |0,n-1\rangle\big] \quad : \ n = 1,2,\ldots,\\
|\psi_{0,n}\rangle &= \frac{1}{\sqrt{2}}\big[|1,n+1\rangle - |0,n\rangle\big] \quad : \ n = 0,1,2,\ldots
\end{aligned}$$

- Energy levels (for $n = 0,1,2,\ldots$):

$$E_{1,n} = \hbar\omega\left(n - \frac{1}{2}\right) + \hbar g_0\sqrt{n}, \quad E_{0,n} = \hbar\omega\left(n + \frac{1}{2}\right) - \hbar g_0\sqrt{n+1}.$$

- Interaction-mediated level splitting between states of opposite parity:

$$E_{1,n} - E_{0,n-1} = 2\hbar g_0\sqrt{n}.$$

- The lowest-energy occurrence of such a pair of levels is known as vacuum Rabi splitting: $E_{1,1} - E_{0,0} = 2\hbar g_0$. Neither level is the lowest.

The Jaynes-Cummings model is an important benchmark in quantum chaos studies of the spin-boson model (to be discussed in a later module).

Cold atoms from laser cooling:

Light-induced forces find applications in trapping atoms or molecules and in laser cooling atomic gases (down to μK region). The nK region is reachable when laser cooling is combined with evaporative cooling.

Rough outline of the principle of laser cooling:

- ▷ The atoms to be cooled have a transition between levels that readily absorbs light from a laser tuned to resonance.
- ▷ ν_0: atomic transition frequency.
- ▷ $\nu_{\rm L} = \nu_0 + \delta$: operating frequency of tunable laser.
- ▷ Doppler shift experienced by atoms moving toward laser beam:

$$\nu'_{\rm L} = \nu_{\rm L}\sqrt{\frac{1+v_x/c}{1-v_x/c}} \simeq \nu_{\rm L}\left(1+\frac{v_x}{c}\right) \simeq \nu_0 + \delta + \nu_0\,\frac{v_x}{c}.$$

- ▷ Resonance condition for frequency of Doppler shifted laser light:

$$\delta = -\nu_0\,\frac{v_x}{c} = -\frac{v_x}{\lambda}.$$

- ▷ Desired effect: atoms moving toward laser beam absorb photons with much higher probability.
- ▷ Photon absorption comes with a change in atomic momentum, which is negative for resonant atoms:

$$\Delta p_x = -\hbar k = -\frac{h}{\lambda}.$$

- ▷ Photon absorption is followed by spontaneous emission in random directions or stimulated emission in the direction of the absorbed light. Only cycles involving spontaneous emission contribute to cooling.
- ▷ A large number of absorption/emission cycles at resonance amount to an effective attenuation force acting on atoms moving toward the laser:

$$F_x = \frac{dp_x}{dt} \simeq \frac{\Delta p_x}{2\tau} = -\frac{h}{2\lambda\tau}.$$

  The factor 2 arises from the fact that in strong laser light spontaneous emission compete on a par with stimulated emissions.

- ▷ Laser cooling continually narrows the width of the (thermal) the velocity distribution. That width is related to the variance, $m\langle v_x^2\rangle \sim k_{\rm B}T$.

▷ The effectiveness of laser cooling diminishes toward zero as the width of the velocity distribution approaches the natural width of the atomic transition, i.e. when

$$k_{\mathrm{B}}T \sim h\Delta\nu = \frac{h}{\tau}.$$

Doppler cooling is limited by the radiative lifetime of the transition.

Optical molasses:

A more rigorous and detailed analysis produces the following expression (not derived here) for the attenuation force associated with laser cooling:

$$F_x(I,\Delta,v_x) = -\hbar k \underbrace{\frac{\gamma}{2}\frac{I}{I_s}\left[1+\frac{I}{I_s}+\left(\frac{\Delta+kv_x}{\gamma/2}\right)^2\right]^{-1}}_{R(I,\Delta,v_x)}.$$

- Photon wave number: $k = 2\pi/\lambda$.
- Natural linewidth: $\gamma = 1/\tau$.
- Optical intensity: I.
- Saturation intensity: I_s.
- Measure of detuning: $\Delta = 2\pi\delta$.
- Net absorption rate:[6] $R(I,\Delta,v_x)$.

At low intensity ($I \ll I_s$) the net absorption rate is proportional to the intensity: $R \propto I$. The dependence of R on Δ is a Lorentizian with a Doppler frequency shift, $\Delta = -kv_x$ i.e. $\delta = -v_x/\lambda$.

At high intensity ($I \gg I_s$) the net absorption rate is determined by the radiative lifetime alone: $R \to \gamma/2 = 1/2\tau$. The factor of 2 introduces the same correction due to stimulated emissions as encountered before.

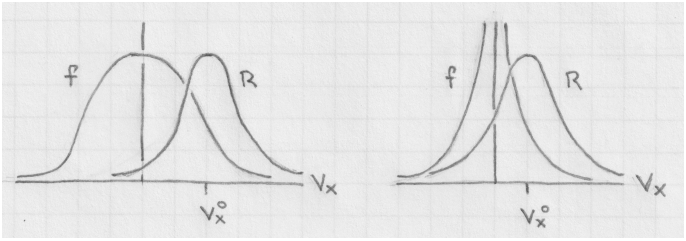
The cooling effectiveness of this design diminishes as T decreases.

The velocity distribution $f(v_x,T)$ is well known to be a Gaussian with a width that narrows as the the temperature drops. However, the width of $R(I,\Delta,v_x)$ does not.

Fewer atoms atoms are being decelerated by the laser beam, while an increasing number of atoms are being accelerated as shown in the plot below.

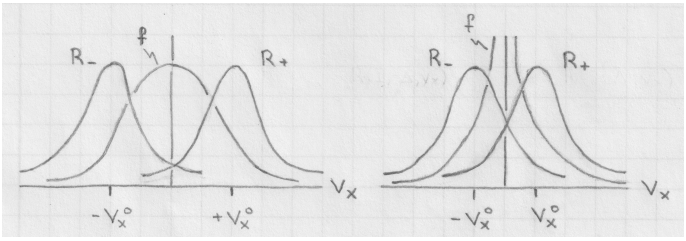
[6] Corrected for (ineffective) stimulated emission.

The velocity distribution $f(v_x, T)$, centered at $v_x = 0$ is broad at high T (left) and narrow at low T (right). The profile of the net absorption rate $R(I, \Delta, v_x)$ is plotted for fixed $\Delta = -kv_x^0$.



This limitation of laser cooling can be pushed to lower T by a change of design that uses two laser beams irradiating the gas from opposite directions.

The plots below make it plain that cooling remains more effective down to lower temperatures. The absorption profiles shown are $R_\pm \doteq R(I, \mp kv_x^0, v_x)$.



The attenuating force in the new design becomes

$$F_x(I, \Delta, v_x) = -\hbar k \big[R(I, \Delta, v_x) - R(I, -\Delta, v_x).$$

At low T, when most atoms are very slow, it is justified the expand the above expression in powers of v_x. The leading term,

$$F_x \rightsquigarrow -\alpha v_x, \quad \alpha = -\frac{8\hbar k^2 \Delta}{\gamma} \left[\frac{I/I_s}{\big[1 + I/I_s + (2\Delta/\gamma)^2\big]^2} \right],$$

represents a standard damping force, always directed opposite to the velocity of the atom as in motion through a viscous medium. Hence the name *optical molasses* used for this effect.

Doppler limit for laser cooling:

It is possible to identify a limiting temperature achievable via laser cooling of any design on the basis of the Doppler effect.

The limiting cooling temperature $T_{\rm DL}$ can be inferred from a balance energy transfers taking place during the process:

– Rate of energy removed via effective damping:

$$\left(\frac{dE}{dt}\right)_{\downarrow} = F_x v_x = -\alpha\langle v_x^2\rangle, \quad \alpha \overset{I/I_s\ll 1}{\longrightarrow} -\frac{8\hbar k^2\Delta}{\gamma}\frac{I/I_s}{[1+4\Delta^2/\gamma^2]^2}.$$

Maximum damping is realized for $\Delta = -\gamma/2$.

– Heat produced during atom/photon interaction:

$$\left(\frac{dE}{dt}\right)_{\uparrow} = \frac{1}{2m}\frac{d}{dt}\langle p_x^2\rangle, \quad \langle p_x^2\rangle = 2N(\hbar k)^2, \quad N = 2Rt.$$

Absorption rate:

$$R(I,\Delta,v_x) \overset{|kv_x|\ll\Delta}{\longrightarrow} \frac{\gamma}{2}\frac{I/I_s}{1+I/I_s+4\Delta^2/\gamma^2} \overset{I/I_s\ll 1}{\longrightarrow} \frac{\gamma}{2}\frac{I/I_s}{1+4\Delta^2/\gamma^2}.$$

$$\Rightarrow \left(\frac{dE}{dt}\right)_{\uparrow} = \frac{\hbar^2k^2\gamma}{m}\frac{I/I_s}{1+4\Delta^2/\gamma^2}.$$

At the Doppler limit, the two rates are in balance:

$$\left(\frac{dE}{dt}\right)_{\downarrow} + \left(\frac{dE}{dt}\right)_{\uparrow} = 0 \quad \Rightarrow \langle v_x^2\rangle\frac{8\hbar k^2\Delta}{\gamma}\frac{I/I_s}{[1+4\Delta^2/\gamma^2]^2} = \frac{\hbar^2k^2\gamma}{m}\frac{I/I_s}{1+4\Delta^2/\gamma^2}.$$

$$\Rightarrow m\langle v_x^2\rangle = \frac{\hbar\gamma^2}{8\Delta}[1+4\Delta^2/\gamma^2] \overset{\Delta\to-\gamma/2}{\longrightarrow} \frac{\hbar\gamma}{2}.$$

Equipartition: $m\langle v_x^2\rangle = k_{\rm B}T_{\rm DL}$.

Doppler limit: $k_{\rm B}T_{\rm DL} = \dfrac{\hbar\gamma}{2} = \dfrac{\hbar}{2\tau}$.