Relativity II [lln25]

Earlier we have identified key differences between situations where nonrelativistic (NR) mechanics reigns and situations where the deviating predictions of special relativity (SR) become significant.

[lln16] For both kinds of situations we can identify events as points with four coordinates: one time coordinate and three space coordinates.

The coordinate transformation of an event as observed from inertial reference frames in relative motion is the *Galilei transformation* for NR situations and the (more general) *Lorentz transformation*, also valid in SR situations.

The Galilei transformation leaves (i) time intervals and (ii) distances (in 3D space) between simultaneous events invariant. The Lorentz transformation leaves distances in 4D spacetime invariant.

Spacetime:

Coordinate 4-vector (contravariant x^μ and covariant x_μ):

$$x^\mu \doteq \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad x_\mu \doteq \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ct \\ x \\ y \\ z \end{pmatrix}.$$

Distance in spacetime (two renditions):[1]

$$(ds)^2 \doteq dx^\mu dx_\mu = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} \begin{pmatrix} -cdt \\ dx \\ dy \\ dz \end{pmatrix} = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2,$$

$$\begin{aligned}(ds)^2 \doteq g_{\mu\nu} dx^\mu dx^\nu &= \Big(cdt, dx, dy, dz\Big) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}, \\ &= -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2\end{aligned}$$

where $g_{\mu\nu}$ is the metric tensor (with signature $-+++$).

The first rendition is a scalar product between two vectors. The second rendition involves two matrix multiplications or two contractions of a rank-4 tensor.

[1]Any repeated spacetime index is understood to be summed over.

Lorentz transformation matrix:

The Lorentz transformation is a linear coordinate transformation in 4-dimensional spacetime between inertial frames in relative motion.

[lln16] Without loss of generality we assume that frame $\mathcal{F}'$ moves with velocity $\mathbf{v} = v\hat{\mathbf{i}}$ relative to frame $\mathcal{F}$.

Transformation matrix:

$$\mathbf{\Lambda} = \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \beta \doteq \frac{v}{c}, \quad \gamma \doteq \frac{1}{\sqrt{1-\beta^2}}.$$

$$\mathbf{\Lambda}^{-1} = \Lambda_{\mu}{}^{\nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\Lambda_{\mu}{}^{\nu}$ and $\Lambda^{\mu}{}_{\nu}$ are symmetric and mutually inverse matrices:

$$\begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lorentz transformation: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ i.e.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma ct - \beta\gamma x \\ \gamma x - \beta\gamma ct \\ y \\ z \end{pmatrix}.$$

[lln16] Consistency with the earlier rendition is evident:

$$t' = \gamma(t - vx/c^2), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z.$$

Equivalent transformation: $x'_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu}$.

Lorentz transformation of 4-vectors and invariant scalars:

We recall that worldlines are strings of events in spacetime associated with a localized physical object. Spacetime positions are 4-vectors.

[lln16] Infinitesimal displacements along worldlines in two reference frames of the same string of events are related via Lorentz transformation, here stated in two equivalent renditions:

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}, \quad dx'_{\mu} = \Lambda_{\mu}{}^{\sigma} dx_{\sigma}.$$

The spacetime distance is an invariant under a Lorentz transformation:

$$(ds)^2 \doteq dx_{\mu} dx^{\mu} = -c^2 (dt)^2 + (dx)^2 + (dy)^2 + (dz)^2.$$

$$\Rightarrow \; dx'_{\mu} dx'^{\mu} = \Lambda_{\mu}{}^{\sigma} dx_{\sigma} \Lambda^{\mu}{}_{\nu} dx^{\nu} = dx_{\sigma} dx^{\nu} \underbrace{\Lambda_{\mu}{}^{\sigma} \Lambda^{\mu}{}_{\nu}}_{\delta^{\sigma}{}_{\nu}} = dx_{\nu} dx^{\nu}.$$

The Lorentz invariant $(ds)^2$, which can be positive or negative, determines
[lln16] the nature of the relation between events:

- Events connected by $(\Delta s)^2 < 0$ are related time-like. They are in the same position at different times for some reference frame, which facilitates a causal relation.
- Events connected by $(\Delta s)^2 > 0$ are related space-like. They are simultaneous and spatially separated for some reference frame, which rules out a causal relation.

Relativistic kinematics and dynamics, including electrodynamics, employ further 4-vectors. They all Lorentz transform in the same way between frames in relative motion.

Associated with each 4-vector is a Lorentz invariant akin to the spacetime distance $(ds)^2$.

However, not all dynamics can be described by 4-vectors and scalars, which are rank-1 and rank-0 tensors, respectively. We shall see that the electromagnetic field, in particular, is represented by a rank-2 tensor.

Kinematics:

A particle is at rest in frame $\mathcal{F}'$.

Frame $\mathcal{F}'$ moves relative to $\mathcal{F}$ with velocity $\mathbf{u}$.

Position increment of particle in frame $\mathcal{F}$: $d\mathbf{x} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$.

Time increment in frame $\mathcal{F}$: dt.

Velocity of particle in $\mathcal{F}$: $\mathbf{u} = \dfrac{d\mathbf{x}}{dt} = u_x\,\hat{\mathbf{i}} + u_y\,\hat{\mathbf{j}} + u_z\,\hat{\mathbf{k}}$.

Proper time $\tau = t'$ is a Lorentz invariant scalar.

Lorentz invariant displacement in spacetime:

$$(ds')^2 = (ds)^2 \quad \Rightarrow \; -(cd\tau)^2 = (d\mathbf{x})^2 - (cdt)^2$$

$$\Rightarrow \; cd\tau = \sqrt{c^2(dt)^2 - (d\mathbf{x})^2} \quad \Rightarrow \; d\tau = dt\,\sqrt{1 - u^2/c^2}.$$

Construction of the velocity 4-vector from the displacement 4-vector dx^μ and the proper-time scalar $d\tau$:

$$dx^\mu = \begin{pmatrix} cdt \\ u_x dt \\ u_y dt \\ u_z dt \end{pmatrix} \quad \Rightarrow \; \eta^\mu \doteq \begin{pmatrix} \eta^0 \\ \eta^1 \\ \eta^2 \\ \eta^3 \end{pmatrix} = \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}} \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}.$$

Lorentz invariant associated with velocity 4-vector is a universal constant:

$$\eta^\mu \eta_\mu = \frac{-c^2 + u^2}{1 - u^2/c^2} = -c^2.$$

The Lorentz transformation of the velocity 4-vector,

$$\eta'^\mu = \Lambda^\mu{}_\nu \eta^\nu,$$

produces the addition rules for the components u_x, u_y, u_x, [lex91][lex219]

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}, \quad u'_y = \frac{u_y/\gamma}{1 - u_x v/c^2}, \quad u'_z = \frac{u_z/\gamma}{1 - u_x v/c^2},$$

[lln16] more directly than earlier the derivation. The time derivative in the construction of η^μ uses the proper time τ, which is a Lorentz invariant scalar.

Energy and momentum:

The rest mass m is a Lorentz invariant scalar.

Construction of the momentum 4-vector from the velocity 4-vector and the rest-mass scalar:

$$p^\mu \doteq \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = m\eta^\mu = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix},$$

composed of relativistic energy (scalar) and momentum (3-vector) familiar from [lln16],

$$E = m\eta^0 c = \frac{mc^2}{\sqrt{1-u^2/c^2}}, \quad \mathbf{p} = m\boldsymbol{\eta} = \frac{m\mathbf{u}}{\sqrt{1-u^2/c^2}}.$$

The Lorentz, transformation of the momentum 4-vector,

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu,$$

ensures that energy and momentum conservation is preserved between frames of reference.

Rest energy: $E_0 = mc^2$ (scalar equivalent to rest mass).

From the Lorentz invariant associated with the momentum 4-vector we construct the relativistic energy-momentum relation as follows:

$$p'_\mu p'^\mu = p_\mu p^\mu \quad \Rightarrow \; E_0^2/c^2 = E^2/c^2 - p^2 \quad \Rightarrow \; E = \sqrt{m^2c^4 + p^2c^2}.$$

Nonrelativistic limit: $E = mc^2\sqrt{1+\dfrac{p^2}{m^2c^2}} \stackrel{p\ll mc}{\longrightarrow} mc^2 + \dfrac{p^2}{2m}$.

The first term (rest energy) has no impact on nonrelativistic dynamics. The second term (kinetic energy) is a key quantity in nonrelativistic dynamics.

Ultrarelativistic limit: $E = \sqrt{m^2c^4 + p^2c^2} \stackrel{p\gg mc}{\longrightarrow} pc.$

Photons (particles without rest mass, traveling at the speed of light) are a realization of the ultrarelativistic limit.

[lln24] The photon energy and momentum are established in quantum mechanics:

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}, \quad \frac{\omega}{|\mathbf{k}|} = c.$$

Dynamics:

The first postulate of special relativity asserts that the laws of physics are the same in all inertial frames. This postulate is satisfied by equations of motion expressed by quantities that Lorentz transform like tensors.

[lln16]

- Rank-0 tensors (scalars) are Lorentz invariants.
- Rank-1 tensors (4-vectors) transform like $a'^{\mu} = \Lambda^{\mu}{}_{\nu} a^{\nu}$.
- Rank-2 tensors transform like $A'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} A^{\alpha\beta}$.

In the context of Newtonian mechanics, the second law is fundamental and the work-energy theorem a corollary.

- Newton's second law: $\dfrac{d\mathbf{p}}{dt} = \mathbf{F}$.
- Work-energy theorem: $\dfrac{dE}{dt} = \mathbf{F} \cdot \mathbf{u}$.

The terms on the left can be assembled into a 4-vector as the derivative of the momentum 4-vector p^{μ} with respect to the (scalar) proper time τ, while force 4-vector K^{μ} is constructed from the terms on the right as follows:

$$\frac{1}{c}\frac{dE}{d\tau} = \frac{1}{c}\frac{dE/dt}{d\tau/dt} = \frac{\mathbf{F} \cdot \mathbf{u}/c}{\sqrt{1 - u^2/c^2}} \doteq K^0,$$

$$\frac{d\mathbf{p}}{d\tau} = \frac{d\mathbf{p}/dt}{d\tau/dt} = \frac{\mathbf{F}}{\sqrt{1 - u^2/c^2}} \doteq \mathbf{K} = \begin{pmatrix} K^1 \\ K^2 \\ K^3 \end{pmatrix}.$$

Covariant form of equation of motion: $\dfrac{dp^{\mu}}{d\tau} = K^{\mu}$.

The force 4-vector K^{μ} goes by the name of Minkowski force.

Whereas all Minkowski forces are 4-vectors, in some cases that 4-vector is the result of a tensor contraction.

Tensor contractions include the inner product of two tensors (involving the summation over one common index.) If the two tensors are of rank n and m, respectively, the contraction produces a tensor of rank $n + m - 2$.

The invariant scalar $a^{\mu} a_{\mu}$ associated with a 4-vector a^{μ} is the contraction of the (rank-2) tensor $a^{\mu} a_{\nu}$ constructed as a tensor product of two 4-vectors.

Lorentz force and electromagnetic field tensor:

Expressing the equation of motion for a particle with charge q subject to the Lorentz force in covariant form naturally leads to a representation of the electric field $\mathbf{E}$ and and the magnetic field $\mathbf{B}$ in the form of a rank-2 tensor.

Lorentz force: $\mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$ (3-vector).

Construction of the Lorentz-force 4-vector:

$$K^0 = \frac{q\mathbf{E}\cdot\mathbf{u}/c}{\sqrt{1-u^2/c^2}} = q\boldsymbol{\eta}\cdot\frac{\mathbf{E}}{c}, \quad \mathbf{K} = \frac{q(\mathbf{E}+\mathbf{u}\times\mathbf{B})}{\sqrt{1-u^2/c^2}} = q\eta^0\,\frac{\mathbf{E}}{c} + q\boldsymbol{\eta}\times\mathbf{B}.$$

– One ingredient is the velocity 4-vector introduced earlier:

$$\eta^\mu = \frac{1}{\sqrt{1-u^2/c^2}}\begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}.$$

– The magnetic force $q\mathbf{u}\times\mathbf{B}$ is perpendicular to $\mathbf{u}$. Therefore, it does not contribute to K^0.

– Lorentz force 4-vector as tensor contraction: $K^\mu = q\eta_\nu F^{\mu\nu}$.

– Electromagnetic field tensor: $F^{\mu\nu} \doteq \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$.

– Temporal component:
$K^0 = q\big(\eta_0 F^{00} + \eta_j F^{0j}\big) = q\eta_j\,\frac{E^j}{c} = q\boldsymbol{\eta}\cdot\frac{\mathbf{E}}{c}$.

– Spatial components:[2]
$K^i = q\big(\eta_0 F^{i0} + \eta_j F^{ij}\big) = q\eta^0\,\frac{E_i}{c} + q\epsilon_{ijk}\eta^j B^k = q\eta^0\,\frac{\mathbf{E}}{c} + q\boldsymbol{\eta}\times\mathbf{B}$.

Covariant form of the equation of motion: $\dfrac{dp^\mu}{d\tau} = q\eta_\nu F^{\mu\nu}$.

This relation features two invariant scalars (q and τ), two 4-vectors (p^μ and η_ν), and one rank-2 tensor ($F^{\mu\nu}$). It involves one contraction of the rank-3 tensor $q\eta_\sigma F^{\mu\nu}$.

[2]The Levi-Civita tensor ϵ_{ijk} vanishes unless all indices have different values. We have $\epsilon_{ijk} = +1$ if their order is cyclic (123, 231, 312) and $\epsilon_{ijk} = -1$ if it is anticyclic.

Lorentz transformation of electromagnetic field tensor:

The principle of relativity enforced for various situations of charge and current configurations produces the following evidence.

[lam27] Field components in frame $\mathcal{F}'$ moving with velocity $v\hat{\mathbf{i}}$ relative to frame $\mathcal{F}$: [lex95]

$$E'_x = E_x, \quad E'_y = \gamma(E_y - vB_z), \qquad E'_z = \gamma(E_z + vB_y),$$
$$B'_x = B_x, \quad B'_y = \gamma(B_y + vE_z/c^2), \quad B'_z = \gamma(B_z - vE_y/c^2).$$

The inverse transformation interchanges primed and unprimed field components and replaces v by $-v$.

Transformation relation for the electromagnetic field tensor:[3]

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}.$$

The (linear) Lorentz transformation mixes electric and magnetic fields as it mixes space and time coordinates and as it mixes momentum components with energy.

Dual electromagnetic field tensor:

$$G^{\mu\nu} \doteq \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}.$$

Duality relation between field tensors: $G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Completely antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ is zero if not all indices are different and $+1$ (-1) for even (odd) permutations of 0123.

Both field tensors feature in a covariant formulation of Maxwell's equations.

Duality here refers to a symmetry of electromagnetism. The duality transformation $F^{\mu\nu} \to G^{\mu\nu}$ is a symmetry transformation, which leaves Maxwell's equations invariant.

Maxwell's equations also contain sources, namely the charge density ρ and the current density $\mathbf{J}$. The sources are also in need of a covariant representation (to be discussed below.)

[3] Given that the fields are functions of space and time, the transformation, $x'^\mu = \Lambda^\mu{}_\lambda x^\lambda$, of the coordinate 4-vector is implied.

Partial derivatives Lorentz transformed:

The fundamental laws of classical electrodynamics – Maxwell's equations for the fields and sources and the continuity equation for the sources – involve partial derivatives with respect to time and space coordinates.[4]

– Gauss's law for the electric field: $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$,

– Gauss's law for the magnetic field: $\nabla \cdot \mathbf{B} = 0$,

– Faraday's law: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$,

– Ampère's law: $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$,

– Continuity equation: $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$.

One direct demonstration of the Lorentz invariance of these laws requires relations between partial derivatives with respect to coordinates of frames in relative motion.

Lorentz transformed differentials of spacetime coordinates:

$$dx' = \gamma(dx - vdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma\left(dt - \frac{v}{c^2}\,dx\right)$$

$$\Rightarrow \; dx = \gamma(dx' + vdt'), \quad dy = dy', \quad dz = dz', \quad dt = \gamma\left(dt' + \frac{v}{c^2}\,dx'\right).$$

Partial derivatives transformed by use of chain rule:

$$\frac{\partial}{\partial x'} = \left(\frac{\partial x}{\partial x'}\right)\frac{\partial}{\partial x} + \left(\frac{\partial t}{\partial x'}\right)\frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right),$$

$$\frac{\partial}{\partial y'} = \left(\frac{\partial y}{\partial y'}\right)\frac{\partial}{\partial y} = \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial z'} = \left(\frac{\partial z}{\partial z'}\right)\frac{\partial}{\partial z} = \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial t'} = \left(\frac{\partial t}{\partial t'}\right)\frac{\partial}{\partial t} + \left(\frac{\partial x}{\partial t'}\right)\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right).$$

[4] We have shown earlier [lln15] that Maxwell's equations are only self-consistent if the sources satisfy the continuity equation.

Current 4-vector and continuity equation:

Evidence for the transformation properties of the charge density ρ and the current density $\mathbf{J}$ if frame $\mathcal{F}'$ moves with velocity $v\hat{\mathbf{i}}$ relative to frame $\mathcal{F}$:

[lam27]

$$\rho' = \gamma\left(\rho - \frac{\beta}{c}\,J_x\right), \quad J'_x = \gamma(J_x - \beta c\rho), \quad J'_y = J_y, \quad J'_z = J_z.$$

- Charge density is enhanced on account of length contraction.
- Current density is enhanced on account of time dilation.
- The charge density in a current-carrying conductor is modified because length contraction differs for positive and negative charge carriers.
- A moving charged object contributes to the current density an amount proportional to its velocity and enhanced by time dilation.

Current 4-vector: $J^\mu = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}, \quad J'^\mu = \Lambda^\mu{}_\nu J^\nu.$

Continuity equation in covariant form,

$$\partial_\mu J^\mu \doteq \frac{\partial J^\mu}{\partial x^\mu} = 0 \quad \Rightarrow \quad \frac{\partial c\rho}{\partial(ct)} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0 \quad \Rightarrow \; \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}.$$

Its construction from two 4-vectors, J^μ and x^μ, guarantees its Lorentz invariance.

Lorentz invariance of continuity equation demonstrated differently:

$$\begin{aligned}
\frac{\partial\rho'}{\partial t'} + \nabla'\mathbf{J}' = \gamma\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial x}\right)\left(\rho - \frac{\beta}{c}\,J_x\right) \\
+ \gamma\left(\frac{\partial}{\partial x} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\gamma(J_x - \beta c\rho) + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \\
= \frac{\partial\rho}{\partial t}\underbrace{(\gamma^2 - \beta^2\gamma^2)}_{1} + \frac{\partial J_x}{\partial x}\underbrace{(-\beta^2\gamma^2 + \gamma^2)}_{1} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \\
= \frac{\partial\rho}{\partial t} + \nabla\mathbf{J} = 0.
\end{aligned}$$

Maxwell's equations in covariant form:

We have introduced the (rank-2) electromagnetic field tensor $F^{\mu\nu}$ and its dual tensor $G^{\mu\nu}$ earlier when we established the Lorentz force as a 4-vector.

Maxwell's equations in covariant form are two relations involving both tensors representing the fields and the current 4-vector representing the sources:

$$\partial_\nu F^{\mu\nu} \doteq \frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \partial_\nu G^{\mu\nu} \doteq \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0.$$

The first equation, which involves sources, is a representation of Gauss's law for the electric field ($\mu = 0$) and Ampère's law ($\mu = 1, 2, 3$).

– Gauss's law for electric field:
$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \mu_0 J^0 \quad \Rightarrow \quad \frac{1}{c}\frac{\partial E_x}{\partial x} + \frac{1}{c}\frac{\partial E_y}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial z} = \mu_0 c\rho \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}.$$

– Ampère's law: $\dfrac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \mu = 1, 2, 3 \quad \Rightarrow \quad \nabla \times \mathbf{B} = \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}.$

$$\mu = 1: -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x \;\Rightarrow\; \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0\epsilon_0 \frac{\partial E_x}{\partial t} + \mu_0 J_x$$

$$\mu = 2: -\frac{1}{c^2}\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} = \mu_0 J_y \;\Rightarrow\; \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0\epsilon_0 \frac{\partial E_y}{\partial t} + \mu_0 J_y$$

$$\mu = 3: -\frac{1}{c^2}\frac{\partial E_z}{\partial t} - \frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x} = \mu_0 J_y \;\Rightarrow\; \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0\epsilon_0 \frac{\partial E_z}{\partial t} + \mu_0 J_z$$

The second equation, which does not involve sources, is a representation of Gauss's law for the magnetic field ($\mu = 0$) and Faraday's law ($\mu = 1, 2, 3$).

– Gauss's law for magnetic field:
$$\frac{\partial G^{0\nu}}{\partial x^\nu} = 0 \quad \Rightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0.$$

– Faraday's law: $\dfrac{\partial G^{\mu\nu}}{\partial x^\nu} = 0, \quad \mu = 1, 2, 3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}.$

$$\mu = 1: \quad -\frac{1}{c}\frac{\partial B_x}{\partial t} - \frac{1}{c}\frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial E_y}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}$$

$$\mu = 2: \quad -\frac{1}{c}\frac{\partial B_y}{\partial t} + \frac{1}{c}\frac{\partial E_z}{\partial x} - \frac{1}{c}\frac{\partial E_x}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

$$\mu = 3: \quad -\frac{1}{c}\frac{\partial B_z}{\partial t} - \frac{1}{c}\frac{\partial E_y}{\partial x} + \frac{1}{c}\frac{\partial E_x}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}$$

Lorentz invariance of Maxwell's equations:

The covariant formulation of Maxwell's equations presented in the previous section guarantees their Lorentz invariance due to their construction from rank-1 and rank-2 tensors.

Alternatively, we can demonstrate the Lorentz invariance of each Maxwell equation separately in the way we accomplished the task earlier for the continuity equation satisfied by the sources ρ and $\mathbf{J}$.

$$\begin{aligned}
\nabla' \cdot \mathbf{B}' &= \frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'} \\
&= \gamma \left(\frac{\partial}{\partial x} + \frac{\beta}{c} \frac{\partial}{\partial t} \right) B_x + \frac{\partial}{\partial y} \gamma \left(B_y + \frac{\beta}{c} E_z \right) + \frac{\partial}{\partial z} \gamma \left(B_z - \frac{\beta}{c} E_y \right) \\
&= \gamma \underbrace{\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right)}_{\nabla \cdot \mathbf{B} = 0} + \gamma \frac{\beta}{c} \underbrace{\left(\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)}_{\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right)_x = 0} = 0.
\end{aligned}$$

$$\begin{aligned}
\nabla' \cdot \mathbf{E}' &= \frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} \\
&= \gamma \left(\frac{\partial}{\partial x} + \frac{\beta}{c} \frac{\partial}{\partial t} \right) E_x + \frac{\partial}{\partial y} \gamma \left(E_y - \beta c B_z \right) + \frac{\partial}{\partial z} \gamma \left(E_z + \beta c B_y \right) \\
&= \gamma \underbrace{\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)}_{\nabla \cdot \mathbf{E}} + \gamma \beta c \underbrace{\left[\frac{1}{c^2} \frac{\partial E_x}{\partial t} - \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \right]}_{\left(\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} \right)_x} \\
&= \gamma \Big[\nabla \cdot \mathbf{E} - \beta c \mu_0 J_x \Big] = \frac{\gamma}{\epsilon_0} \left[\rho - \frac{\beta}{c} J_x \right] = \frac{\rho'}{\epsilon_0}.
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial \mathbf{B}'}{\partial t'} + \nabla' \times \mathbf{E}' \right)_x &= \frac{\partial B'_x}{\partial t'} + \frac{\partial E'_z}{\partial y'} - \frac{\partial E'_y}{\partial z'} \\
&= \gamma \left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial x} \right) B_x + \frac{\partial}{\partial y} \gamma (E_z + \beta c B_y) - \frac{\partial}{\partial z} \gamma (E_y - \beta c B_z) \\
&= \gamma \beta c \underbrace{\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right)}_{\nabla \cdot \mathbf{B} = 0} + \gamma \underbrace{\left(\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)}_{\left(\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right)_x = 0} = 0.
\end{aligned}$$

The invariance of Faraday's law for the other two Cartesian components and of Ampère's law are demonstrated similarly. [lex194]

4-vector potential:

Definition: $A^\nu = \begin{pmatrix} \Phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix}.$

Relation between potentials and fields in covariant form:

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1}$$

▷ $F^{01} = \dfrac{E_x}{c} = -\dfrac{1}{c}\dfrac{\partial A_x}{\partial t} - \dfrac{1}{c}\dfrac{\partial \Phi}{\partial x}, \quad F^{02} = \dfrac{E_y}{c} = -\dfrac{1}{c}\dfrac{\partial A_y}{\partial t} - \dfrac{1}{c}\dfrac{\partial \Phi}{\partial y},$

$F^{03} = \dfrac{E_z}{c} = -\dfrac{1}{c}\dfrac{\partial A_z}{\partial t} - \dfrac{1}{c}\dfrac{\partial \Phi}{\partial z} \quad \Rightarrow \mathbf{E} = -\nabla\Phi - \dfrac{\partial \mathbf{A}}{\partial t}.$

▷ $F^{12} = B_z = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}, \quad F^{13} = -B_y = \dfrac{\partial A_z}{\partial x} - \dfrac{\partial A_x}{\partial z},$

$F^{23} = B_x = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \quad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}.$

▷ The remaining off-diagonal elements duplicate these results.

▷ The diagonal elements are identically zero.

Implications for the structure of Maxwell's equations in covariant form as established earlier:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0.$$

▷ The dependence (1) of $F^{\mu\nu}$ on A^ν guarantees that tensor $G^{\mu\nu}$ as inferred from the duality relation satisfies the second (homogeneous) relation.

▷ The first (inhomogeneous) relation can be transformed as follows:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left[\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right] = \frac{\partial}{\partial x_\mu}\left[\frac{\partial A^\nu}{\partial x^\nu}\right] - \frac{\partial}{\partial x_\nu}\left[\frac{\partial A^\mu}{\partial x^\nu}\right] = \mu_0 J^\mu.$$

[lln16] ▷ The Lorenz gauge condition implies

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t} \quad \Rightarrow \frac{\partial A^\nu}{\partial x^\nu} = 0.$$

▷ The first relation becomes an inhomogeneous wave equation:

$$\frac{\partial^2 A^\mu}{\partial x_\nu \partial x^\nu} = \underbrace{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]}_{\text{D'Alembertian } \Box} A^\mu = -\mu_0 J^\mu$$

Electromagnetic wave observed in moving frames:

Linearly polarized plane wave traveling in the positive x-direction.

Views from frame $\mathcal{F}$ and from frame $\mathcal{F}'$ traveling with velocity $\mathbf{v} = v\,\hat{\mathbf{i}}$ relative to frame $\mathcal{F}$:

$$\mathbf{E}(\mathbf{x},t) = E_0 \cos(kx - \omega t)\,\hat{\mathbf{j}}, \quad \mathbf{B}(\mathbf{x},t) = \frac{E_0}{c}\cos(kx - \omega t)\,\hat{\mathbf{k}};$$

$$\mathbf{E}'(\mathbf{x}',t') = E_0' \cos(k'x' - \omega' t')\,\hat{\mathbf{j}}, \quad \mathbf{B}'(\mathbf{x}',t') = \frac{E_0'}{c}\cos(k'x' - \omega' t')\,\hat{\mathbf{k}}.$$

Transformation of 4-vector $(\omega/c, \mathbf{k})$: (here with $k_x = k,\ k_y = k_z = 0$)

$$kx - \omega t = k\gamma(x' + vt') - \omega\gamma(t' + vx'/c^2) = k'x' - \omega' t'.$$

Collect coefficients of x' and t':

$$\Rightarrow\ k' = \gamma(k - v\omega/c^2), \quad \omega' = \gamma(\omega - vk).$$

Doppler effect for electromagnetic wave: use $\omega' = ck'$ in the first equation or $\omega = ck$ in the second equation:

$$\omega' = \gamma(1 - v/c)\omega = \sqrt{\frac{1 - v/c}{1 + v/c}}\,\omega.$$

For $v > 0$ ($v < 0$) we observe a red shift (blue shift) in frame $\mathcal{F}'$.

The transformation of the field tensor produces amplitude changes as well:

$$E_0' = \gamma(E_0 - vB_0) = \gamma(1 - v/c)E_0 = \sqrt{\frac{1 - v/c}{1 + v/c}}\,E_0,$$

$$B_0' = \gamma(B_0 - (vE_0/c^2) = \gamma(1 - v/c)B_0 = \sqrt{\frac{1 - v/c}{1 + v/c}}\,B_0.$$

A red shift (blue shift) is associated with a decreasing (increasing) amplitude in both the electric and magnetic fields.

Energy-momentum flux tensor:

Earlier we have introduced the momentum 4-vector p^μ of a massive particle and the equation of motion for a charged particle in an electromagnetic field:

$$\frac{dp^\mu}{d\tau} = q\eta_\nu F^{\mu\nu}.$$

Here we generalize this equation to a continuum of charged matter positioned and in moving in an electromagnetic field:

$$\frac{\partial \mathcal{P}^\mu}{\partial \tau} = J_\nu F^{\mu\nu}.$$

– Momentum density 4-vector: $\mathcal{P}^\mu$.

– Current 4-vector (from earlier): J^μ.

– Electromagnetic field tensor: $F^{\mu\nu}$.

– Flux of energy and momentum carried by charged matter: $J_\nu F^{\mu\nu}$.

Continuity equation expressing charge conservation:

$$\frac{\partial J^\mu}{\partial x^\mu} = 0.$$

Continuity equation expressing the conservation of field energy in the absence of sources (charges and currents):

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0.$$

In this relation we postulate a tensor $T^{\mu\nu}$ expressing the flux of energy and momentum pertaining to the electromagnetic field.

The generalization of this continuity equation in the presence of charged matter adds a term which expresses the transfer of energy and momentum between field and matter:

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + J_\nu F^{\mu\nu} = 0.$$

The *energy-momentum flux tensor* of the electromagnetic field, also named *stress-energy-momentum tensor* is constructed as follows:

$$T^{\mu\nu} = \frac{1}{\mu_0}\left[F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right],$$

where $g^{\mu\nu}$ is the metric tensor introduced earlier.

– Temporal component ($\mu = 0$): Poynting theorem

$$\Rightarrow \ \frac{1}{c}\mathbf{J}\cdot\mathbf{E} = -\frac{1}{c}\frac{\partial T^{00}}{\partial t} - \frac{\partial T^{0i}}{\partial x^i}.$$

▷ $\mathbf{J}\cdot\mathbf{E}$: rate at which electric field does work on charged matter,

▷ $T^{00} = u = \frac{1}{2\mu_0}\left[\frac{E^2}{c^2} + B^2\right]$: field energy density,

▷ $cT^{0i} = \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$: Poynting vector.

– Spatial components ($\mu = i = 1, 2, 3$):

$$\Rightarrow \ \rho E^i + \epsilon_{ijk}J^j B^k = -\frac{1}{c}\frac{\partial T^{i0}}{\partial t} - \frac{\partial T^{ij}}{\partial x^j}.$$

▷ $\rho E^i + \epsilon_{ijk}J^j B^k$: force per unit volume acting on charged matter,

▷ $\frac{T^{i0}}{c} = \mathbf{\Pi} = \frac{\mathbf{S}}{c^2}$: momentum density of electromagnetic field,

▷ T^{ij}: momentum flux.

Summary list of Lorentz transformations:

– spacetime coordinates:
$t' = \gamma(t - vx_\parallel/c^2), \quad x'_\parallel = \gamma(x_\parallel - vt), \quad \mathbf{x}'_\perp = \mathbf{x}_\perp.$

– velocity:
$u'_\parallel = \frac{u_\parallel - v}{1 - vu_\parallel/c^2}, \quad \mathbf{u}'_\perp = \frac{\mathbf{u}_\perp}{\gamma(1 - vu_\parallel/c^2)}.$

– energy and momentum:
$E' = \gamma(E - vp_\parallel), \quad p'_\parallel = \gamma(p_\parallel - vE/c^2), \quad \mathbf{p}'_\perp = \mathbf{p}_\perp.$

– electric and magnetic fields:
$E'_\parallel = E_\parallel, \quad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}_\perp),$
$B'_\parallel = B_\parallel, \quad \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \mathbf{v}\times\mathbf{E}_\perp/c^2).$

The inverse transformation interchanges primed and unprimed field components and replaces $v, \mathbf{v}$ by $-v, -\mathbf{v}$, respectively. [lex96][lex146][lex147]