Quantum Optics I [lln24]

Not all optical phenomena can be explained by electromagnetic waves interacting with continuous media. For some, light is a stream of photons.

Levels of quantization:

Interactions of matter and light are described on three levels of quantization:

- *Classical*: Light treated as (classical) electromagnetic wave interacts with atoms treated as (classical) Hertzian dipoles – constituents of electrically polarizable material.
- *Semiclassical*: Atoms with (quantized) electronic energy levels interact with (classical) electromagnetic waves.
- *Quantum*: Atoms with electronic (quantum) level spectrum interact with photons (quantized light).

Many experiments that suggest light quantization (e.g. the photoelectric effect) do not provide conclusive evidence for it. A semiclassical description is often quite adequate.

Quantum optics took off as a separate field of research only around 1970. The table lists landmarks in its development.

Year	Authors	Development
1901	Planck	Theory of black-body radiation
1905	Einstein	Explanation of the photoelectric effect
1909	Taylor	Interference of single quanta
1909	Einstein	Radiation fluctuations
1927	Dirac	Quantum theory of radiation
1956	Hanbury Brown and Twiss	Intensity interferometer
1963	Glauber	Quantum states of light
1972	Gibbs	Optical Rabi oscillations
1977	Kimble, Dagenais, and Mandel	Photon antibunching
1981	Aspect, Grangier, and Roger	Violations of Bell's inequality
1985	Slusher *et al.*	Squeezed light
1987	Hong, Ou, and Mandel	Single-photon interference experiments
1992	Bennett, Brassard *et al.*	Experimental quantum cryptography
1995	Turchette, Kimble *et al.*	Quantum phase gate
1995	Anderson, Wieman, Cornell *et al.*	Bose–Einstein condensation of atoms
1997	Mewes, Ketterle *et al.*	Atom laser
1997	Bouwmeester *et al.*, Boschi *et al.*	Quantum teleportation of photons
2002	Yuan *et al.*	Single-photon light-emitting diode

[image from Fox 2014]

Blackbody radiation:

Electromagnetic radiation inside a cavity in thermal equilibrium with the walls has an energy density which only depends on temperature T.

The spectral distribution of that energy density cannot be explained by a theory based on classical electrodynamics.

A grandcanonical ensemble of photons ($\epsilon = \hbar\omega = cp$, $\mathbf{p} = \hbar\mathbf{k}$, spin $s = 1$, bosonic, purely transverse) does the trick (it's a long story told elsewhere).

Density of states (e.g. modes of standing waves in a cube of volume V):

$$D(\epsilon) = g\,\frac{4\pi V}{h^3c^3}\,\epsilon^2, \quad (g = 2: \text{ transverse polarizations}).$$

Average occupation number of energy level: $\langle n_\epsilon \rangle = \dfrac{1}{e^{\beta\epsilon} - 1}$.

Number of photons with energies between ϵ and $\epsilon + d\epsilon$:

$$dN(\epsilon) = \langle n_\epsilon \rangle D(\epsilon) d\epsilon = \frac{8\pi V \epsilon^2}{h^3c^3}\frac{1}{e^{\beta\epsilon} - 1} d\epsilon.$$

Spectral density inside cavity: [use $dN(\epsilon) = V dn(\omega)$ and $\epsilon = \hbar\omega$]:

$$\frac{dn(\omega)}{d\omega} = \frac{\hbar}{V}\frac{dN(\epsilon)}{d\epsilon} = \frac{\omega^2}{\pi^2c^3}\frac{1}{e^{\beta\hbar\omega} - 1}.$$

Spectral energy density inside cavity: $\hbar\omega dn = u(\omega)d\omega$.

$$\Rightarrow \quad u(\omega) = \underbrace{\frac{\omega^2}{\pi^2c^3}}_{(a)}\underbrace{\frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}}_{(b)} = \frac{d(\omega)}{e^{\beta\hbar\omega} - 1}, \quad d(\omega) = \frac{\hbar\omega^3}{\pi^2c^3}, \tag{1}$$

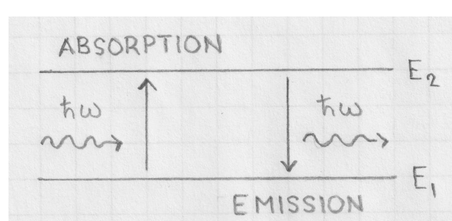
(a) spectral density of modes,

(b) energy content of single mode.

Einstein coefficients:

The first theory of radiative transitions in atoms was proposed by Einstein in the earliest days quantum physics.

Consider transitions between two atomic states: $E_2 - E_1 = \hbar\omega$ with level occupancies N_1, N_2 and level degeneracies g_1, g_2.
Assume radiation with unknown spectral energy density $u(\omega)$.



Distinguish three types of transitions:

– Spontaneous emission: $\dfrac{dN_2}{dt} = -A_{21}N_2 \quad \Rightarrow \; N_2(t) = N_2(0)e^{-t/\tau}$.

– Absorption: $\dfrac{dN_1}{dt} = -B_{12}u(\omega)N_1$.

– Stimulated emission: $\dfrac{dN_2}{dt} = -B_{21}u(\omega)N_2$.

Spontaneous emission is governed by a radiative lifetime, $\tau = 1/A_{21}$, of one type or other (to be further discussed later).

Condition of detailed balance:

$$\frac{d}{dt}(N_1 + N_2) = 0 \quad \Rightarrow \; B_{12}u(\omega)N_1 = A_{21}N_2 + B_{21}u(\omega)N_2.$$

Condition of thermal equilibrium: $\dfrac{N_2}{N_1} = \dfrac{g_2}{g_1}\, e^{-\beta\hbar\omega}$.

The spectral energy density extracted from the two conditions,

$$u(\omega) = \frac{g_2 A_{21} e^{-\beta\hbar\omega}}{g_1 B_{12} - g_2 B_{21} e^{-\beta\hbar\omega}},$$

is consistent with (1) if we set $g_1 B_{12} = g_2 B_{21}$ and $A_{21} = B_{21} d(\omega)$. It suffices to calculate just one of three Einstein coefficients from first principles.

Radiative transition rates:

The general framework for this task is time-dependent perturbation theory.

Fermi's golden rule: $W_{1\to 2} = \dfrac{2\pi}{\hbar}\, |M_{12}|^2 g_2(\hbar\omega)$,

▷ $W_{1\to 2}$: rate of transitions between electronic states,

▷ M_{12}: matrix element for specific type of interaction,

▷ $g_2(\omega)$: density of final states.

The interaction between light (electric and magnetic fields) and atoms (electric charges, spins, dipole moments) has many parts.

The strongest contribution comes from electric dipole interaction (E1), subdominant contributions from magnetic dipole interaction (M1) and electric quadrupole interaction (E2). Transition matrix elements are subject to selection rules related to symmetries, which are discussed elsewhere.

Linewidth and lineshape:

Radiation is, in general, not perfectly monochromatic. Spectral lines are broadened and vary in shape.

It is useful to distinguish two classes of broadening mechanisms:

- *homogeneous* broadening affects all sources equally,
- sources of *inhomogeneous* broadening vary in linewidths.

Lineshapes are characterized by (normalized) lineshape functions $g_\omega(\omega)$.

The sources of line broadening are manifold. A few examples are briefly discussed in the following.

- *Lifetime broadening* (homogeneous):

  The source of this type of broadening (also named natural or radiative broadening) is rooted in the uncertainty principle, $\Delta E \Delta t \gtrsim \hbar$.

  Spontaneous emissions are exponentially distributed in time. The average lifetime of an excited state is named τ_{LT}. The resulting lineshape is Lorentzian (Fourier transform of exponential distribution):

$$g_\omega^{\mathrm{L}}(\omega) = \frac{\Delta\omega_{\mathrm{L}}/2\pi}{(\omega-\omega_0)^2 + (\Delta\omega_{\mathrm{L}}/2)^2}, \quad \Delta\omega_{\mathrm{L}} = \frac{1}{\tau_{\mathrm{L}}}.$$

- *Doppler broadening* (inhomogeneous):

  The inhomogeneity of this type of broadening is associated with the velocity distribution in gases, even though it is spatially homogeneous.

  The Doppler effect singles out the direction toward the observer. To leading order we can write $\omega = \omega_0(1 \pm v_x/c)$.

  The Maxwell velocity distribution is Gaussian in nature. It thus produces a Gaussian lineshape [tex63]:

$$g_\omega^{\mathrm{D}}(\omega) = \sqrt{\frac{mc^2}{2\pi k_{\mathrm{B}} T \omega_0^2}} \exp\left(-\frac{mc^2(\omega-\omega_0)^2}{2k_{\mathrm{B}} T \omega_0^2}\right), \quad \frac{\Delta\omega_{\mathrm{D}}}{2\omega_0} = \sqrt{\frac{(2\ln 2)k_{\mathrm{B}} T}{mc^2}}.$$

- *Collisional broadening*:

  Interatomic collisions tend to trigger radiative processes. This shortens the lifetimes of excited states in gases significantly.

  The mean collision time in a dilute gas is inversely proportional to pressure [tex70]. Hence the alternative name of pressure broadening.

  Whereas the causes of broadening are distributed homogeneously the trigger probability is inhomogeneous.

– *Line broadening in solids*:

There are two principal causes for this type of broadening, the first homogeneous and the second inhomogeneous.

(i) Non-radiative transitions (involving phonons) increase the Einstein coefficient A_{21} associated with spontaneous emissions, which shortens the lifetime of excited states.

(ii) Environmental broadening is associated with structural inhomogeneities (impurities, dislocations, ...).

Photon statistics:

The interpretation of a beam of light as a traveling wave (w) of electric and magnetic fields or a stream (s) of photons are complementary in Bohr's sense.

Both interpretations describe the same amount of energy and momentum traveling at the speed of light in the same direction.

The same beam exhibits a diffraction pattern in one experiment [using (w)] or produces a sequence of signals in a counter [using (s)].

In the following, we focus on the question of how the photon-counting statistics depends on the make-up of the wave.

A few words on photon detection:

– Detection devices for photons include photomultiplier tubes (PMT), avalanche photodiodes (APD), and more sophisticated instruments.
– Since all photons travel at the same speed, the spatial photon distribution in the beam translates into an equal distribution of arrival times at the counter.
– Detectors have a recovery time (dead time after registering a signal). Photons too close in sequence are not resolved as separate events. The problem can be partially mitigated by a reduction of beam power.

Relevant specifications for the case of a perfectly coherent monochromatic beam as realized by a linearly polarized plane wave.

▷ ω: angular frequency [rad/s],
▷ $\hbar\omega$: photon energy [J],
▷ E_0: electric-field amplitude [V/m],
▷ $B_0 = E_0/c$: magnetic-field amplitude [T],

- ▷ $I = \frac{1}{2}\epsilon_0 E_0^2$: intensity [$\mathrm{Jm^{-2}s^{-1}}$] [J]
- ▷ A: cross sectional area of detector [$\mathrm{m^2}$],
- ▷ $P = IA$: power of signal picked up by detector [J/s],
- ▷ $\Phi = \dfrac{P}{\hbar\omega}$: photon flux [$\mathrm{s^{-1}}$],
- ▷ T: counting time [s] (time interval selected in experiment),
- ▷ $n(T)$: average number of counts registered,
- ▷ $\eta = \dfrac{n(T)}{\Phi T} = \dfrac{n(T)\hbar\omega}{PT}$: quantum efficiency ($0 \leq \eta \leq 1$),
- ▷ $R = \dfrac{n(T)}{T} = \eta\Phi = \dfrac{\eta P}{\hbar\omega}$: count rate [$\mathrm{s^{-1}}$].

Photon statistics is classified on the basis of characteristics of a distribution: the mean value $\langle n\rangle$ and the variance $\langle\langle n^2\rangle\rangle \doteq \langle n^2\rangle - \langle n\rangle^2$.

The mean photon count $\langle n\rangle$ is controlled by the intensity of the beam. What determines the variance $\langle\langle n^2\rangle\rangle$? The intrinsic photon statistics of the beam or the nature of the photodetection process?

Probability distribution $P(n)$ of photon counts:

$$\sum_{n=0}^{\infty} P(n) = 1, \;\; \sum_{n=0}^{\infty} nP(n) = \langle n\rangle, \;\; \sum_{n=0}^{\infty} n^2P(n) = \langle n^2\rangle, \;\; \langle\langle n^2\rangle\rangle \doteq \langle n^2\rangle - \langle n\rangle^2.$$

Photon statistics is divided into regimes according to the relation between mean and variance:

- Poisson statistics: $\langle\langle n^2\rangle\rangle = \langle n\rangle$,
- super-Poisson statistics: $\langle\langle n^2\rangle\rangle > \langle n\rangle$,
- sub-Poisson statistics: $\langle\langle n^2\rangle\rangle < \langle n\rangle$.

Poisson statistics has a benchmarks degree of fluctuations in relation to the average photon count. Super-Poisson statistics has a higher degree of fluctuations and sub-Poisson statistics a lower degree.

The consensus is that Poisson statistics and super-Poisson statistics are compatible with a classical interpretation of light, whereas sub-Poisson statistics is a true quantum feature.

There are alternative and equally compelling ways to classify a stream of photons, e.g. by the attributes of intensity correlations (to be discussed in a later module).

Light with Poisson statistics:

Coherent light produces a stream of photons with a count sequence characterized by the (one-parameter) Poisson distribution [lex149]:[1]

$$P(n) = \frac{a^n}{n!}\, e^{-a}, \quad \langle n \rangle = \langle\langle n^2 \rangle\rangle = a.$$

This result follows if we assume that single photons arrive at the detector in a completely random sequence.

Consider a steady beam with intensity such that the average time interval between successive photon detections is τ.

Next we define (without prejudice) two probabilities to be determined:

- Probability that the interval is between t and $t + dt$: $f(t)dt$.
- Probability that the interval is larger than t: $P_0(t) = \int_t^\infty dt' f(t')$.
- Differential relation: $f(t) = -\frac{dP_0}{dt}$.
- Normalizations: $P_0(0) = 1, \quad P_0(\infty) = 0 \quad \Rightarrow \int_0^\infty dt\, f(t) = 1$.
- Set mean value: $\langle t \rangle \doteq \int_0^\infty dt\, t f(t) = \tau$.

For the determination of $P_0(t)$ we examine events A, B, $\bar{B}$ and start the clock just when a photon has arrived at the detector:

- event A: the next photon has not arrived by time t,
- event B: a photon arrives between times t and $t + dt$.
- event $\bar{B}$: complement of event B.

Consequences of postulated randomness:

▷ $P(AB) = P(A)P(B)$ (statistical independence),

▷ $P(B) = cdt$ for tiny dt with c to be determined,

▷ $P_0(t + dt) = P(A\bar{B}) = P(A)P(\bar{B}) = P_0(t)[1 - cdt]$.

$$\Rightarrow \frac{d}{dt}P_0(t) = -cP_0(t) \;\Rightarrow\; P_0(t) = e^{-ct} \;\Rightarrow\; f(t) = ce^{-ct}.$$

[1]The connection between Poisson statistics and coherence will be demonstrated later.

Determine c from the known mean value: $\langle t\rangle = \tau \;\Rightarrow\; c = 1/\tau$.

$$\Rightarrow\; P_0(t) = e^{-t/\tau}, \quad f(t) = \frac{1}{\tau}e^{-t/\tau}, \quad \langle\langle t^2\rangle\rangle = \tau^2.$$

The time intervals between photon detections are exponentially distributed.

The probability $P_n(t)$ that n photons have been counted by time t can be determined recursively from $P_{n-1}(t)$ as follows:

The probabilities $f(t')dt'$ that the first photon arrives between t' and $t'+dt'$ and $P_0(t-t')$ that no further photon arrives until time t factorize. The probability that exactly one photon arrives until time t then becomes,

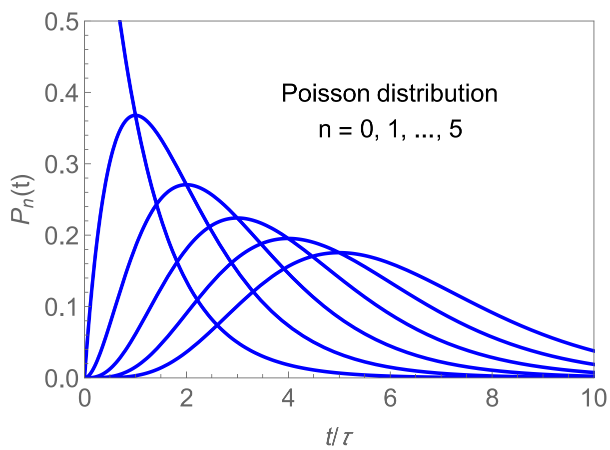
$$P_1(t) = \int_0^t dt' f(t')P_0(t-t') = \frac{t}{\tau}e^{-t/\tau}.$$

This recursion relation generalizes into

$$P_n(t) = \int_0^t dt' f(t')P_{n-1}(t-t') \tag{2}$$

and thus produces the Poisson distribution:

$$P_n(t) = \frac{(t/\tau)^n}{n!}e^{-t/\tau}, \quad \langle n\rangle = \langle\langle n^2\rangle\rangle = \frac{t}{\tau}.$$



The maximum of $P_n(t)$ is located at $t = n\tau$.

Boundary conditions and normalization condition:

$$P_n(0) = \delta_{n,0}, \quad P_n(\infty) = 0 \;:\; n = 0,1,2,\ldots; \quad \sum_{n=0}^{\infty} P_n(t) = 1 \;:\; t \geq 0.$$

Note that the function $f(t)$ is a continuous probability distributions, whereas the functions $P_n(t)$ constitute a discrete probability distribution.

Light with Pascal statistics:

We have seen that it takes a completely random sequence of photons to bring the variance up to the level of the mean. How can the fluctuations be made yet stronger with the same average number of photons?

Incoherent light produces a stream of photons with a count sequence characterized by a distribution $P(n)$ whose variance exceeds the mean:

$$\langle\langle n^2\rangle\rangle > \langle n\rangle.$$

One possibility to augment statistical fluctuations as captured in the variance is by allowing the photons to arrive in randomly sequenced bunches.

Perfectly coherent light is free of fluctuations in a classical description. Partially coherent light (named chaotic) is subject to intensity fluctuations.

Blackbody radiation (described earlier) is a well-characterized form of incoherent light.

How many photons of energy $\hbar\omega$ in a steady stream emerging from a cavity at temperature T arrive at the detector in a given time?

The probability of counting n photons is characterized by the (one-parameter) Pascal distribution [lex150]:

$$P(n) = \gamma^n \Big/ \sum_{n=0}^{\infty} \gamma^n = (1-\gamma)\gamma^n, \quad \gamma \doteq e^{-\beta\hbar\omega}.$$

The mean and the variance of this distribution encapsulate the super-Poisson character of the statistics [lex150]:

$$\langle n\rangle = \frac{\gamma}{1-\gamma}, \quad \langle\langle n^2\rangle\rangle = \frac{\gamma}{(1-\gamma)^2} = \langle n\rangle + \langle n\rangle^2.$$

The mean is consistent with the energy content of a single mode of blackbody radiation discussed earlier:

$$\langle n\rangle = \frac{\gamma}{1-\gamma} = \frac{1}{e^{\beta\hbar\omega}-1},$$

With the following setting we assume a steady stream of photons where one photon arrives in time τ on average:

$$\langle n\rangle = \frac{\gamma}{1-\gamma} = \frac{t}{\tau}, \quad \langle\langle n^2\rangle\rangle = \frac{t}{\tau}\left(1+\frac{t}{\tau}\right).$$

The difference $\langle\langle n^2\rangle\rangle - \langle n\rangle$ is not only positive (signature of super-Poisson statistics), it is also steadily increasing with detection time.

The ensuing probability that no photon arrives in time t,

$$P_0(t) = 1 - \gamma = \frac{1}{1 + t/\tau},$$

decays more slowly than in a stream with the same average where single photons arrive randomly (Poisson statistics).

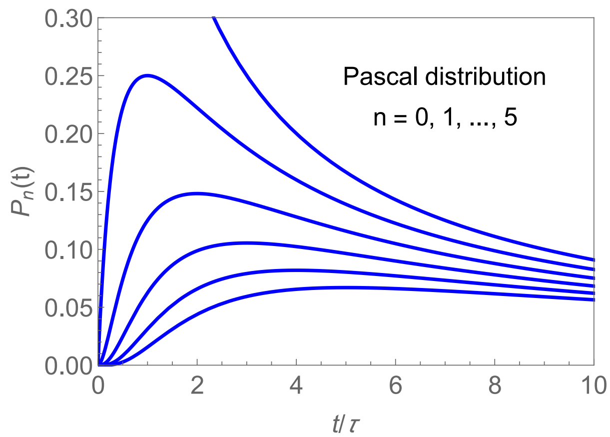
The distribution of intervals between detections,

$$f(t) = -\frac{dP_0}{dt} = \frac{1/\tau}{(1 + t/\tau)^2},$$

predicts large intervals with a higher probability than a random stream of single photons would. The mean time interval between detections diverges:

$$\langle t \rangle = \int_0^\infty dt\, t f(t) = \infty.$$

On the other hand, the relation, $\langle n \rangle = t/\tau$, states that one photon hits the detector in time τ on average. Some photons must arrive in bunches, as might be the result of stimulated emission.



The functions $P_n(t)$ inferred from $P(n)$ are

$$P_n(t) = \frac{(t/\tau)^n}{(1 + t/\tau)^{n+1}}.$$

The maximum of $P_n(t)$ is again located at $t = n\tau$. The proportionality between t and n is indicative of the steady stream. The augmented fluctuations relative to Poisson statistics is indicated by the broader peaks.

The two kinds of normalization conditions still hold:

$$P_n(0) = \delta_{n,0}, \quad P_n(\infty) = 0 \quad : \; n = 0, 1, 2, \ldots; \quad \sum_{n=0}^{\infty} P_n(t) = 1 \quad : \; t \geq 0.$$

The integral recursion relation (2) established previously for the $P_n(t)$ of the Poisson distribution does not hold for the Pascal distribution. Two conditions that went into (2) are not satisfied:

{1} The function $f(t)$ now represents the probability distribution of intervals between detection. Each detection may involve multiple photons.

{2} The probabilities $f(t')dt'$ and $P_{n-1}(t-t')$ do not necessarily factorize.

Here we briefly explore if something can be learned from the integral on the right-hand side of (2).

The result for $n = 1$ is intriguing [lex180]:

$$\int_0^t dt' f(t') P_0(t-t') = \frac{t/\tau}{(1+t/\tau)(2+t/\tau)} + \frac{2}{(2+t/\tau)^2} \ln\left(1+\frac{t}{\tau}\right).$$

The second term rises linearly from zero at $t = 0$, goes through a smooth maximum and approaches zero faster than the first term does for $t \to \infty$. The second term might be attributable to point {2}.

The first term is readily attributable to point {1} once we recognize that the following relation holds:

$$\sum_{n=1}^{\infty} 2^{-n} P_n(t) = \frac{t/\tau}{(1+t/\tau)(2+t/\tau)}.$$

For $t \gg \tau$ the following integral relation thus holds in good approximation for the Pascal distribution:

$$\int_0^t dt' f(t') P_0(t-t') \simeq \sum_{n=1}^{\infty} 2^{-n} P_n(t).$$

This relation is compatible with the reinterpretation of the function $f(t)$ as a probability distribution for the time intervals between detection events which involve multiple photons.

Earlier we showed the following key difference between Poisson statistics and Pascal statistics:

– In a stream of coherent light with an intensity of one photon per time τ, the average time interval between detections is τ again.

– In a stream of thermal light (of one wavelength) with the same intensity, the average time interval between detections diverges due to a preponderance of large intervals.

Light with super-Poisson statistics:

The goal in this section is to model a super-Poisson distribution which interpolates between the Poisson and Pascal distributions, i.e. between coherent light and light (of one frequency) emitted from a cavity.

An interpolating expression is readily constructed for $P_0(t)$:

$$P_0(t) = \left(1 + \frac{t}{m\tau}\right)^{-m} \longrightarrow \begin{cases} \dfrac{1}{1+t/\tau} & : \; m = 1 \quad \text{(Pascal)}, \\ e^{-t/\tau} & : \; m = \infty \quad \text{(Poisson)}. \end{cases}$$

The interpolating super-Poisson distribution thus depends on a discrete parameter m with the range of the natural numbers.

Next we search for an attribute shared by the two limiting cases from which we can generate the $P_n(t)$ for $n = 1, 2, \ldots$.

The integral recursion relation (2) previously established for the Poisson distribution does not hold for the Pascal distribution as we have examined earlier. Therefore, it is of no use for what we have in mind.

However, the following differential recursion relation can be shown to hold for both limiting cases [lex179]:

$$(n+1)P_{n+1}(t) = nP_n(t) - tP_n'(t), \quad n = 1, 2, \ldots$$

We can use it to generate the $P_n(t)$ for $n = 1, 2, \ldots$ and any value of the interpolation parameter m. The result can be cast in the form [lex177]:

$$P_n(t) = \left(1 + \frac{1}{m}\right)\left(1 + \frac{2}{m}\right)\cdots\left(1 + \frac{n-1}{m}\right)\frac{1}{n!}\left(\frac{t}{\tau}\right)^n \left(1 + \frac{t}{m\tau}\right)^{-(m+n)}.$$

The two limiting cases are readily recovered from this expression [lex177]:

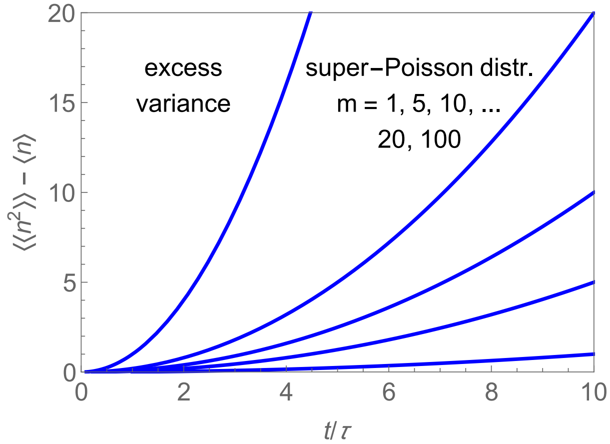
$$\lim_{m\to 1} P_n(t) = \frac{(t/\tau)^n}{(1+t/\tau)^{n+1}}, \qquad \lim_{m\to\infty} P_n(t) = \frac{(t/\tau)^n}{n!} e^{-t/\tau}.$$

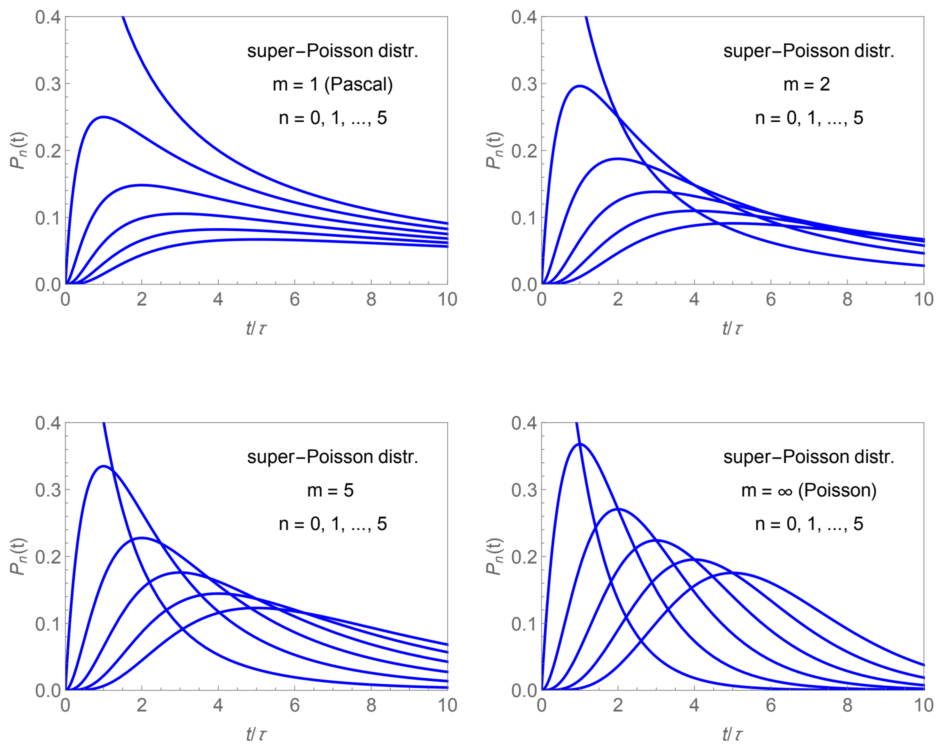
The interpolation verifiably conserves the normalization [lex178]. The mean value stays exactly at the value of the Pascal and Poisson limits [lex178]:

$$\langle n \rangle = \frac{t}{\tau} \quad : \text{ independent of } m.$$

The variance interpolates smoothly between the two limits [lex178]:

$$\langle\langle n^2 \rangle\rangle = \frac{t}{\tau}\left(1 + \frac{t}{m\tau}\right).$$





The maximum of $P_n(t)$ is located at time $t = n\tau$ for any value of the interpolation parameter m.

It remains to be seen how well this super-Poisson model photon statistics matches data for incoherent light of any kind.

The distribution of time intervals between detection depends on the interpolation parameter as follows:

$$f(t) \doteq -\frac{dP_0}{dt} = \frac{1/\tau}{(1+t/m\tau)^{1+m}} \longrightarrow \begin{cases} \dfrac{1/\tau}{(1+t/\tau)^2} & : \; m = 1 \quad \text{(Pascal)}, \\ \dfrac{1}{\tau} e^{-t/\tau} & : \; m = \infty \quad \text{(Poisson)}. \end{cases}$$

For the average detection interval and its variance we infer the following expressions:

$$\langle t \rangle = \frac{m}{m-1}\,\tau, \quad \langle\langle t^2 \rangle\rangle = \frac{m^3}{(m-1)^2(m-2)}\,\tau^2.$$

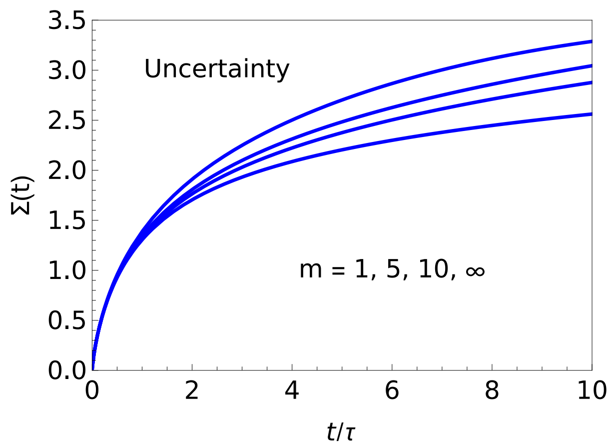
The former diverges for $m \leq 1$, the latter for $m \leq 2$. Higher moments $\langle t^n \rangle$ converge only if the exponent n falls below the interpolation parameter m:

$$\langle t^n \rangle = \frac{\Gamma(m-n)\Gamma(1+n)}{\Gamma(m)}, \quad n < m.$$

A measure for the uncertainty of a probability distribution is justified in [tsc7] and defined as follows:

$$\Sigma(t) = -\sum_n P_n(t) \ln P_n(t).$$

When applied to the super-Poisson probability distribution for selected values of the interpolation parameters, the uncertainty of Σ as a function of scaled detection time t/τ yields the results shown.



All curves rise from zero as expected. At any nonzero time, coherent light ($m = \infty$) has the least uncertainty and thermal light ($m = 1$) the most uncertainty regarding the exact photon count.

Light with sub-Poisson statistics:

A stream of photons with a count sequence characterized by a distribution $P(n)$ whose variance falls below the mean,

$$\langle\langle n^2\rangle\rangle < \langle n\rangle,$$

defies any classical description. Sub-Poisson statistics implies some degree of regularity in the stream of photons.

A stream of photons with sub-Poisson statistics can be modeled on the basis of Erlang distributions. Here we present one case to be generalized later.

We begin with the distribution of time intervals between photon detections and infer the distribution for detecting no photon until time t:[2]

$$f(t) \doteq \frac{4t}{\tau^2}\, e^{-2t/\tau} \quad \Rightarrow \; P_0(t) \doteq \int_t^\infty dt' f(t') = \left(1 + \frac{2t}{\tau}\right) e^{-2t/\tau}.$$

Normalization, mean, and variance:

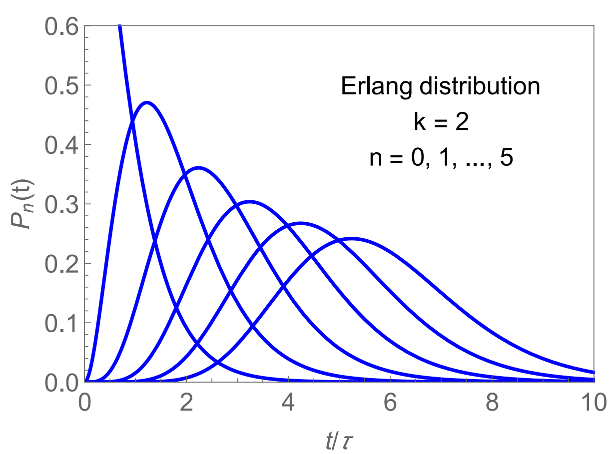
$$\int_0^\infty dt\, f(t) = 1, \quad \langle t\rangle = \tau, \quad \langle\langle t^2\rangle\rangle = \frac{\tau^2}{2}.$$

The integral recursion relation for the Poisson statistics,

$$P_n(t) = \int_0^t dt' f(t') P_{n-1}(t - t'),$$

is assumed to remain valid. It determines the remaining probabilities [lex151]:

$$P_n(t) = \left[\frac{(2t/\tau)^{2n}}{(2n)!} + \frac{(2t/\tau)^{(2n+1)}}{(2n+1)!}\right] e^{-2t/\tau}.$$



[2]The function $f(t)$ is known as an Erlang distribution with index $k = 2$. The case $k = 1$ is represented by the exponential distribution (Poisson statistics). Erlang distributions for arbitrary index k will be further explored below.

The two kinds of normalization conditions are again satisfied:

- ▷ $P_n(0) = \delta_{n,0}, \quad P_n(\infty) = 0 \quad : \; n = 0, 1, 2, \ldots$
- ▷ $\sum_{n=0}^{\infty} P_n(t) = 1 \quad : \; t \geq 0.$

The mean and the variance confirm the sub-Poisson nature for $t \gg \tau$:

$$\langle n \rangle = \frac{1}{4}\left[e^{-4t/\tau} - 1 + \frac{4t}{\tau}\right] \rightsquigarrow \begin{cases} 2\left(\dfrac{t}{\tau}\right)^2 & : \; t \ll \tau, \\ \dfrac{t}{\tau} - \dfrac{1}{4} & : \; t \gg \tau, \end{cases}$$

$$\langle\langle n^2 \rangle\rangle = \frac{1}{16}\left[1 - e^{-8t/\tau} + \frac{8t}{\tau}\left(1 - 2e^{-4t/\tau}\right)\right] \rightsquigarrow \begin{cases} 2\left(\dfrac{t}{\tau}\right)^2 & : \; t \ll \tau, \\ \dfrac{t}{2\tau} + \dfrac{1}{16} & : \; t \gg \tau. \end{cases}$$

Statistics of photon number states:

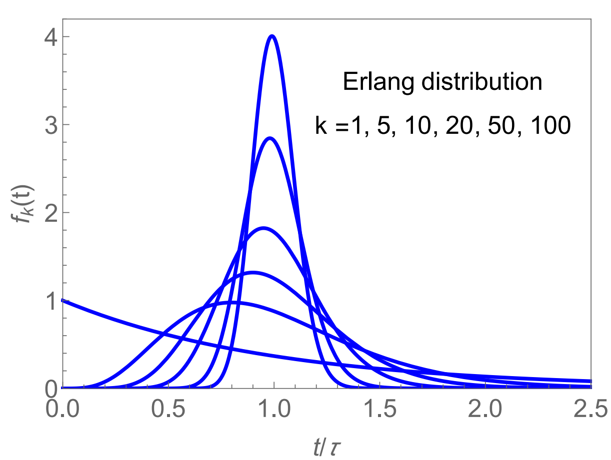
The limiting case of sub-Poisson statistics with mean value $\langle n \rangle = t/\tau$ and variance $\langle\langle n^2 \rangle\rangle = 0$ is known as photon number state.

We invoke the Erlang distribution with index k to demonstrate this limit:

$$f^{(k)}(t) \doteq \frac{(k/\tau)^k t^{k-1}}{(k-1)!}\, e^{-kt/\tau} \quad \stackrel{k\to\infty}{\longrightarrow} \quad \delta(t - \tau).$$

It is normalized, has steady mean and (sub-Poisson) variance:

$$\int_0^\infty dt\, f^{(k)}(t) = 1, \quad \langle t \rangle = \tau, \quad \langle\langle t^2 \rangle\rangle = \frac{\tau^2}{k}.$$



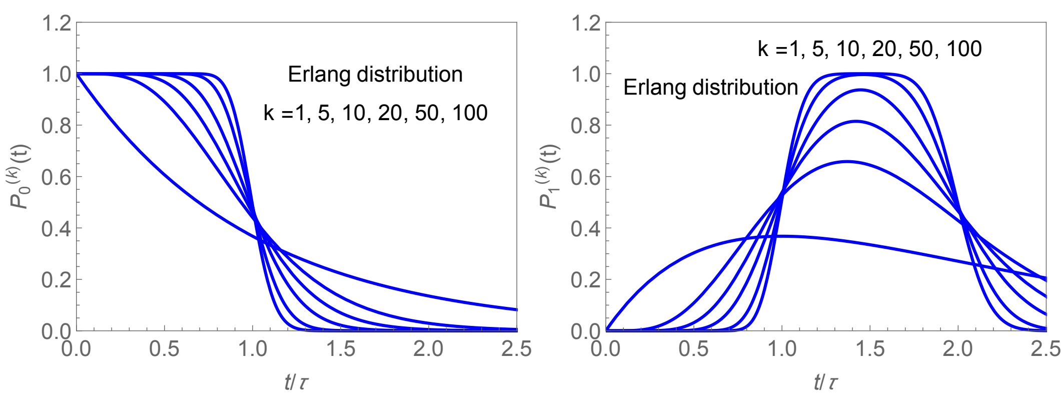
The goal is to generate the distribution $P_n^{(k)}(t/\tau)$ for arbitrary k including the limit $k \to \infty$. We have already discussed two cases earlier. The pattern seen for $k = 1, 2, 3$ suggests the following generalization [lex152]:[3]

$$P_n^{(k)}(t) = \left[\sum_{m=0}^{k-1} \frac{(kt/\tau)^{kn+m}}{(kn+m)!}\right] e^{-kt/\tau} = \frac{\Gamma(kn+k, kt/\tau)}{\Gamma(kn+k)} - \frac{\Gamma(kn, kt/\tau)}{\Gamma(kn)}.$$

The numerical evidence inferred from this expression suggests that is that the following relation between mean and variance asymptotically for $t \gg \tau$:

$$\langle n \rangle \rightsquigarrow \frac{t}{\tau}, \quad \langle\langle n^2 \rangle\rangle \rightsquigarrow \frac{t}{k\tau}.$$

In the plots of $P_0^{(k)}(t)$ (left) and $P_1^{(k)}(t)$ (right) we recognize the growing degree of regularity in the photon sequence as the index k increases.



In the limit $k \to \infty$ all functions $P_n^{(k)}(t)$ become rectangular:

$$P_n^{(\infty)}(t) = \Theta\big(t - n\tau\big) - \Theta\big(t - (n+1)\tau\big).$$

The mean value then becomes a staircase function (integer part of t/τ) and the variance vanishes identically:

$$\langle n \rangle = \text{Floor}(t/\tau), \quad \langle\langle n^2 \rangle\rangle \equiv 0.$$

The difference $\langle\langle n^2 \rangle\rangle - \langle n \rangle$ for finite k is negative (signature of sub-Poisson statistics), and approaches an asymptotic value for large detection times.

The uncertainty Σ (defined earlier) vanishes identically at all t for $P_n^{(\infty)}(t)$.

[3]The incomplete Gamma function is defined as follows: $\Gamma(\alpha, x) \doteq \int_x^\infty dt\, t^{\alpha-1} e^{-t}$. Completion is achieved in the limit $x \to 0$: $\Gamma(\alpha, 0)$ is renamed $\Gamma(\alpha)$. For integer arguments it is equivalent to the factorial: $\Gamma(m) = (m-1)!$