Radiation I [lln19]

Electric charges at rest generate a time-independent electric field. Steady currents generate a time-independent magnetic field.

A charged particle moving at constants velocity produces electric and magnetic fields that move along with the particle in a time-independent pattern.

It takes electric charges in accelerated motion to produce radiation – an electromagnetic wave propagating away from its source. Alternating currents are one realization of electric charges in accelerated motion.

D'Alembert equations for scalar and vector potentials:

We start from Maxwell's equations for the electric field **E** and magnetic field **B** in the presence of ideal sources ρ (charge density) and **J** (current density),

[lln15]

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0},\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t},$$
$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{B}=\mu_0\left(\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right),$$

and their transcription into the D'Alembert equations,

$$-\nabla^2\Phi+\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=\frac{\rho}{\epsilon_0},\quad -\nabla^2\mathbf{A}+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\mu_0\mathbf{J},\tag{1}$$

for the scalar potential Φ and vector potential **A** in the Lorenz gauge and the sources satisfying the continuity equation:

$$\nabla\cdot\mathbf{A}=-\frac{1}{c^2}\frac{\partial\Phi}{\partial t},\qquad \nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}.$$

[lln5][ln12] Earlier analysis for time-independent situations:

- Poisson equations: $-\nabla^2\Phi(\mathbf{x})=\dfrac{\rho(\mathbf{x})}{\epsilon_0},\quad -\nabla^2\mathbf{A}(\mathbf{x})=\mu_0\mathbf{J}(\mathbf{x})$
- Mathematical identity: $\nabla^2\left(\dfrac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=-4\pi\delta(\mathbf{x}-\mathbf{x}'),$
- Scalar potential: $\Phi(\mathbf{x})=\dfrac{1}{4\pi\epsilon_0}\displaystyle\int d^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$
- Vector potential: $\mathbf{A}(\mathbf{x})=\dfrac{\mu_0}{4\pi}\displaystyle\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$
- Conditions for ources: $\nabla\cdot\mathbf{J}=0,\quad \dfrac{\partial\rho}{\partial t}=0.$

Retarded potentials:

Generalization to situations with harmonic time-dependence (artificial from a physical perspective, but useful from a mathematical perspective):

- Oscillating charge density: $\rho(\mathbf{x},t) = \tilde{\rho}(\mathbf{x})e^{-\imath\omega t}$.
- Ansatz for scalar potential: $\Phi(\mathbf{x},t) = \tilde{\Phi}(\mathbf{x})e^{-\imath\omega t}$.
- D'Alembert equation becomes Helmholtz equation:

$$-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad -(\nabla^2 + k^2)\tilde{\Phi}(\mathbf{x}) = \frac{\tilde{\rho}(\mathbf{x})}{\epsilon_0}, \quad k = \frac{\omega}{c}.$$

- Linear (differential) operator: $\mathcal{L} \doteq -(\nabla^2 + k^2)$.
- Green's function associated with Helmholtz equation: [lex111]

$$\mathcal{L}G(\mathbf{x}-\mathbf{x}') = 4\pi\delta(\mathbf{x}-\mathbf{x}') \quad \Rightarrow \; G(\mathbf{x}-\mathbf{x}') = \frac{e^{\imath k|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}.$$

- Solution of Helmholtz equation: $\tilde{\Phi}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' G(\mathbf{x}-\mathbf{x}')\tilde{\rho}(\mathbf{x}')$.

$$\Rightarrow \; \mathcal{L}\tilde{\Phi}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \underbrace{\mathcal{L}G(\mathbf{x}-\mathbf{x}')}_{4\pi(\delta(\mathbf{x}-\mathbf{x}')}\tilde{\rho}(\mathbf{x}') = \frac{\tilde{\rho}(\mathbf{x})}{\epsilon_0}.$$

- Scalar potential inferred from this solution:

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\tilde{\rho}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, e^{-\imath\omega t + \imath k(\mathbf{x}-\mathbf{x}')} = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}.$$

- Retarded time: $-\imath\omega t + \imath k(\mathbf{x}-\mathbf{x}') \doteq -\imath\omega t' \quad \Rightarrow \; t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$.
- Current density: $\mathbf{J}(\mathbf{x},t) = \tilde{\mathbf{J}}(\mathbf{x})e^{-\imath\omega t}$.
- Vector potential: $\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}$.
- Condition for sources: $\nabla \cdot \tilde{\mathbf{J}} = \imath\omega\tilde{\rho}$.

Fourier integrals:

Generalization to situations with unrestricted time evolution.

The superposition of solutions oscillating at different frequencies can be expressed as a Fourier integral.

– Charge density and current density:

$$\rho(\mathbf{x},t) = \int_{-\infty}^{+\infty} d\omega\, \tilde{\rho}(\mathbf{x},\omega) e^{-\imath\omega t} \;\Leftrightarrow\; \tilde{\rho}(\mathbf{x},\omega) = \frac{1}{2\pi}\int dt\, \rho(\mathbf{x},t) e^{\imath\omega t},$$

$$\mathbf{J}(\mathbf{x},t) = \int_{-\infty}^{+\infty} d\omega\, \tilde{\mathbf{J}}(\mathbf{x},\omega) e^{-\imath\omega t} \;\Leftrightarrow\; \tilde{\mathbf{J}}(\mathbf{x},\omega) = \frac{1}{2\pi}\int dt\, \mathbf{J}(\mathbf{x},t) e^{\imath\omega t},$$

– Superposition principle applied to scalar and vector potentials:

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} d\omega\, e^{-\imath\omega t} \int d^3x' e^{\imath k|\mathbf{x}-\mathbf{x}'|}\, \frac{\tilde{\rho}(\mathbf{x}',\omega)}{|\mathbf{x}-\mathbf{x}'|} \\ &\overset{k=\omega/c}{=} \frac{1}{4\pi\epsilon_0}\int d^3x'\, \frac{1}{|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{+\infty} d\omega\, e^{-\imath\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}\tilde{\rho}(\mathbf{x}',\omega) \\ &= \frac{1}{4\pi\epsilon_0}\int d^3x'\, \frac{\rho(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \qquad (2)\end{aligned}$$

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} d\omega\, e^{-\imath\omega t} \int d^3x' e^{\imath k|\mathbf{x}-\mathbf{x}'|}\, \frac{\tilde{\mathbf{J}}(\mathbf{x}',\omega)}{|\mathbf{x}-\mathbf{x}'|} \\ &\overset{k=\omega/c}{=} \frac{1}{4\pi\epsilon_0}\int d^3x'\, \frac{1}{|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{+\infty} d\omega\, e^{-\imath\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}\tilde{\mathbf{J}}(\mathbf{x}',\omega) \\ &= \frac{1}{4\pi\epsilon_0}\int d^3x'\, \frac{\mathbf{J}(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \qquad (3)\end{aligned}$$

[lex112]

– Retardation: $t' = t - \dfrac{|\mathbf{x}-\mathbf{x}'|}{c}$.

– Condition for sources: $\nabla'\cdot\tilde{\mathbf{J}}(\mathbf{x}',t') = -\dfrac{\partial}{\partial t}\rho(\mathbf{x}',t')$.

The potentials Φ and $\mathbf{A}$ at field point $\mathbf{x}$ and time t depend on the sources ρ and $\mathbf{J}$, respectively, at source point $\mathbf{x}'$ and delayed time t', which depends on the distance $|\mathbf{x}-\mathbf{x}'|$.

The usefulness of solutions (2) and (3) of the D'Alembert equations (1) is limited to situations for which the potentials are caused by known sources.

In general, sources and potentials (and the fields derived from the potentials) perform an interactive dance with time-delayed causations both ways.

Radiation from electric dipole:

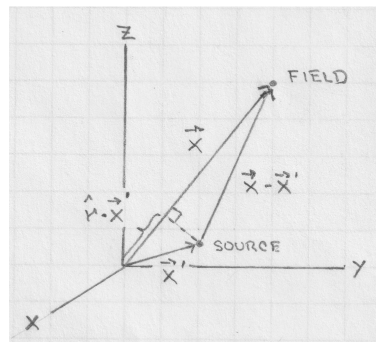
The general integral expressions for the retarded potentials $\Phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ are the basis of antenna theory – a broad area of electrical engineering.

The structure of electric and magnetic fields changes with distance from the source of radiation. The radiation zone – far away from the source – is of primary interest in most applications.

Prototype radiation source: a very localized region of current density $\mathbf{J}(\mathbf{x}', t)$.

Analysis of field expressions at large distances $|\mathbf{x}| \doteq r$ from the source (asymptotic regime, $|\mathbf{x}| \gg \hat{\mathbf{r}} \cdot \mathbf{x}'$).

$$\begin{aligned} \frac{1}{|\mathbf{x} - \mathbf{x}'|} &\simeq \frac{1}{|\mathbf{x}| - \hat{\mathbf{r}} \cdot \mathbf{x}'} \\ &= \frac{1}{|\mathbf{x}|} \frac{1}{1 - \hat{\mathbf{r}} \cdot \mathbf{x}'/|\mathbf{x}|} \\ &\simeq \frac{1}{|\mathbf{x}|} \left(1 + \frac{\hat{\mathbf{r}} \cdot \mathbf{x}'}{|\mathbf{x}|}\right). \end{aligned}$$



Integral expression for asymptotic vector potential:

$$\mathbf{A}(\mathbf{x}, t)_{\text{as}} = \frac{\mu_0}{4\pi r} \int d^3x' \, \mathbf{J}(\mathbf{x}', t_r), \quad t_r \doteq t - \frac{r}{c}.$$

The variation of the retarded time t_r across the source is disregarded. This assumption is justified if the wavelength of the radiation is large compared to the size of the source.

[lln5] In electrostatics, electric dipoles originate in charge densities. In electrodynamics, charge densities $\rho(\mathbf{x}, t)$ and current densities $\mathbf{J}(\mathbf{x}, t)$ are related to
[lln15] each other by the continuity equation.

In consequence, the time-dependence of electric dipoles can be inferred from $\rho(\mathbf{x}, t)$ or from $\mathbf{J}(\mathbf{x}, t)$: [lex113]

$$\mathbf{p}(t) = \int d^3x' \, \mathbf{x}' \rho(\mathbf{x}', t), \quad \frac{d}{dt} \mathbf{p}(t) = \int d^3x' \, \mathbf{J}(\mathbf{x}', t).$$

Asymptotic vector potential from electric dipole moment:

$$\Rightarrow \; \mathbf{A}(\mathbf{x}, t)_{\text{as}} = \frac{\mu_0}{4\pi r} \left. \frac{d\mathbf{p}}{dt} \right|_{t_r}, \quad t_r \doteq t - \frac{r}{c}.$$

Magnetic radiation field from $\mathbf{B}(\mathbf{x},t)_{\rm as} = \nabla\times\mathbf{A}(\mathbf{x},t)_{\rm rad}$.

- Ignore spatial derivatives acting on factors $1/r$. They yield contributions which are negligible in radiation zone.
- Transform curl using chain rule: $\nabla\times\left.\frac{d\mathbf{p}}{dt}\right|_{t_r} = \nabla t_r\times\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}$. [lex114]
- Evaluate gradient: $\nabla t_r = -\frac{\hat{\mathbf{r}}}{c}$.
- Radiation magnetic field: $\mathbf{B}(\mathbf{x},t)_{\rm rad} = -\frac{\mu_0}{4\pi rc}\hat{\mathbf{r}}\times\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}$.

Scalar potential inferred from Lorentz gauge condition: [lex115]

$$\frac{\partial}{\partial t}\Phi = -c^2\nabla\cdot\mathbf{A} \quad\Rightarrow\quad \Phi(\mathbf{x},t)_{\rm rad} = \frac{\mu_0 c}{4\pi r}\,\hat{\mathbf{r}}\cdot\left.\frac{d\mathbf{p}}{dt}\right|_{t_r}.$$

Electric radiation field from $\mathbf{E} = c\mathbf{B}\times\hat{\mathbf{r}}$ or $\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\Phi$: [lex222][lex223]

$$\mathbf{E}(\mathbf{x},t)_{\rm rad} = \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}}\left(\hat{\mathbf{r}}\cdot\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right) - \left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right].$$

The transverse direction of the radiation fields follows from the vanishing scalar products, $\mathbf{E}(\mathbf{x},t)_{\rm rad}\cdot\hat{\mathbf{r}} = 0$ and $\mathbf{B}(\mathbf{x},t)_{\rm rad}\cdot\hat{\mathbf{r}} = 0$.

Notice that the $\sim r^{-1}$ dependence of the (asymptotic) electric radiation field due to a dynamic electric dipole is different from the $\sim r^{-3}$ dependence of the electric field generated by a static electric dipole.

Poynting vector from $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$: [lex116]

$$\mathbf{S}(\mathbf{x},t)_{\rm rad} = \frac{\mu_0/c}{(4\pi r)^2}\left[\left(\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right)^2 - \left(\hat{\mathbf{r}}\cdot\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right)^2\right]\hat{\mathbf{r}}.$$

Notice the radial direction of $\mathbf{S}$ in the radiation zone.

Total power of radiation from $P(r,t) = \oint d\Omega\, r^2\,\hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{x},t)_{\rm rad}$: [lex116]

$$\Rightarrow\ P(t) = \frac{\mu_0}{6\pi c}\left(\left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r}\right)^2.$$

Energy conservation makes this quantity independent of distance.

Hertzian dipole (special case):

Harmonically oscillating electric dipole moment placed at the origin of a spherical coordinate system, oscillating in $\pm z$-direction (North-South at the center of a sphere):

$$\mathbf{p}(t) = p_0 \cos(\omega t)\,\hat{\mathbf{z}}.$$

Magnetic radiation field (directed East-West on the surface of the sphere):

$$\mathbf{B}_{\rm ed}(\mathbf{x},t)_{\rm rad} = -\frac{\mu_0}{4\pi r c}\hat{\mathbf{r}} \times \left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r} = -\frac{\mu_0 \omega^2 p_0}{4\pi r c}\sin\theta\cos(\omega t_r)\,\hat{\boldsymbol{\phi}}.$$

Electric radiation field (directed South-North on the surface of the sphere):

$$\mathbf{E}_{\rm ed}(\mathbf{x},t)_{\rm rad} = c\mathbf{B}_{\rm ed}(\mathbf{x},t)_{\rm rad} \times \hat{\mathbf{r}} = -\frac{\mu_0 \omega^2 p_0}{4\pi r}\sin\theta\cos(\omega t_r)\,\hat{\boldsymbol{\theta}}.$$

Poynting vector (directed radially outward):

$$\mathbf{S}_{\rm ed}(\mathbf{x},t)_{\rm rad} = \frac{1}{\mu_0}\mathbf{E}_{\rm ed}(\mathbf{x},t)_{\rm rad} \times \mathbf{B}_{\rm ed}(\mathbf{x},t)_{\rm rad} = \frac{\mu_0 \omega^4 p_0^2}{(4\pi r)^2 c}\sin^2\theta\cos^2(\omega t_r)\,\hat{\mathbf{r}}.$$

Intensity (average power per unit area transported locally):

$$I_{\rm ed} \doteq \langle|\mathbf{S}_{\rm ed}|\rangle = \frac{\mu_0 \omega^4 p_0^2}{(4\pi r)^2 c}\sin^2\theta\,\underbrace{\langle\cos^2(\omega t_r)\rangle}_{1/2}.$$

Differential power (transported per solid angle):

$$\frac{dP_{\rm av}^{\rm (ed)}}{d\Omega} = r^2\,\hat{\mathbf{r}}\cdot\langle|\mathbf{S}_{\rm ed}|\rangle = \frac{\mu_0 \omega^4 p_0^2}{32\pi^2 c}\sin^2\theta = \frac{3P_{\rm av}^{\rm (ed)}}{8\pi}\sin^2\theta.$$

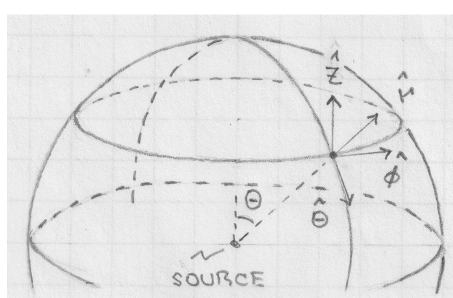
Average power (averaged over directions): [lex117]

$$P_{\rm av}^{\rm (ed)} \doteq 2\pi\int_0^\pi d\theta\sin\theta\,\frac{dP_{\rm av}^{\rm (ed)}}{d\Omega} = \frac{\mu_0 \omega^4 p_0^2}{16\pi c}\underbrace{\int_0^\pi d\theta\sin^3\theta}_{4/3} = \frac{\mu_0 \omega^4 p_0^2}{12\pi c}.$$

$\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\hat{\boldsymbol{\phi}}\sin\theta$

$\hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$

$\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}$



Radiation from magnetic dipole:

A conducting loop of area A carrying an alternating current $I(t)$ emits magnetic dipole radiation.

- Small flat loop of area A in xy-plane at origin of coordinate system.
- Alternating current: $I(t) = I_0 e^{-\imath\omega t}$.
- Electric dipole is absent: $\mathbf{p}(t) \equiv 0$.
- Magnetic dipole moment: $\mathbf{m}(t) = AI_0 e^{-\imath\omega t}\,\hat{\mathbf{z}} = m_0 e^{-\imath\omega t}\,\hat{\mathbf{z}}$.
- The vector $\mathbf{m}(t)$ is oscillating up-down at the center of a sphere.
- Assumption that loop is small: $\sqrt{A} \ll \lambda = 2\pi c/\omega$.
- Vector potential from (2): $\mathbf{A}(\mathbf{x},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int d^3x'\,\dfrac{\mathbf{J}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}$.
- Current density expanded using $|\mathbf{x}-\mathbf{x}'| \simeq r - \mathbf{x}'\cdot\hat{\mathbf{r}}$:

$$\mathbf{J}(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c) = \mathbf{J}(\mathbf{x}', t_r) + \frac{\mathbf{x}'\cdot\hat{\mathbf{r}}}{c}\frac{\partial}{\partial t_r}\mathbf{J}(\mathbf{x}', t_r), \quad t_r = t - \frac{r}{c}.$$

- First integral vanishes if $\rho \equiv 0$ [lex113]: $\displaystyle\int d^3x'\,\mathbf{J}(\mathbf{x}', t_r) = \left.\frac{d\mathbf{p}}{dt}\right|_{t_r} = 0.$ [lex113]

[lln12] - Result from earlier: $\displaystyle\int d^3x'\,(\mathbf{x}'\cdot\hat{\mathbf{r}})\,\mathbf{J}(\mathbf{x}', t_r) = \mathbf{m}(t_r)\times\hat{\mathbf{r}}$.

- Second integral evaluated in steps, using $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\sin\theta$:

$$\mathbf{A}_{\rm md}(\mathbf{x},t)_{\rm rad} = \frac{\mu_0}{4\pi rc}\int d^3x'\,(\mathbf{x}'\cdot\hat{\mathbf{r}})\frac{\partial}{\partial t_r}\mathbf{J}(\mathbf{x}', t_r) = \frac{\mu_0}{4\pi rc}\frac{\partial}{\partial t_r}\mathbf{m}(t_r)\times\hat{\mathbf{r}}$$
$$= -\frac{\imath\omega\mu_0}{4\pi rc}\mathbf{m}(t_r)\times\hat{\mathbf{r}} = -\frac{\imath\mu_0\omega m_0}{4\pi rc}e^{-\imath\omega(t-r/c)}\sin\theta\,\hat{\boldsymbol{\phi}}.$$

Radiation vector potentials generated by electric and magnetic dipoles:

$$\mathbf{A}_{\rm ed}(\mathbf{x},t)_{\rm rad} = -\frac{\imath\mu_0\omega p_0}{4\pi r}e^{-\imath\omega(t-r/c)}\,\hat{\mathbf{z}},$$
$$\mathbf{A}_{\rm md}(\mathbf{x},t)_{\rm rad} = -\frac{\imath\mu_0\omega m_0}{4\pi rc}e^{-\imath\omega(t-r/c)}\underbrace{\sin\theta\,\hat{\boldsymbol{\phi}}}_{\hat{\mathbf{z}}\times\hat{\mathbf{r}}} = \frac{m_0}{cp_0}\mathbf{A}_{\rm ed}(\mathbf{x},t)_{\rm rad}\times\hat{\mathbf{r}}.$$

The relative power of radiation sources originating in electric and magnetic dipoles is controlled by the dimensionless ratio, m_0/cp_0.

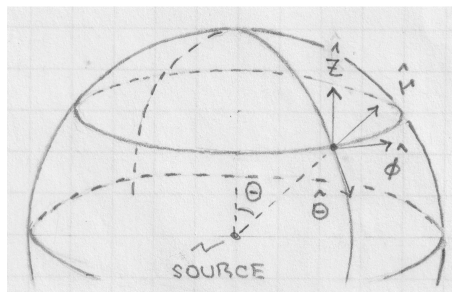
In sources with $p_0 > 0$, the dominant role is likely taken by the electric dipole. The contribution of the magnetic dipole is suppressed by a factor c.

Radiation fields of magnetic dipole:

$$\mathbf{E}_{\rm md}(\mathbf{x},t)_{\rm rad} = \frac{m_0}{cp_0}\mathbf{E}_{\rm ed}(\mathbf{x},t)_{\rm rad}\times\hat{\mathbf{r}} = \frac{\mu_0\omega^2 m_0}{4\pi rc}\sin\theta\cos(\omega t_r)\,\hat{\boldsymbol{\phi}},$$
$$\mathbf{B}_{\rm md}(\mathbf{x},t)_{\rm rad} = \frac{m_0}{cp_0}\mathbf{B}_{\rm ed}(\mathbf{x},t)_{\rm rad}\times\hat{\mathbf{r}} = -\frac{\mu_0\omega^2 m_0}{4\pi rc^2}\sin\theta\cos(\omega t_r)\,\hat{\boldsymbol{\theta}}.$$

$\hat{\boldsymbol{\phi}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$

$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}} = -\hat{\boldsymbol{\phi}}$



Poynting vector (directed radially outward):

$$\mathbf{S}_{\rm md}(\mathbf{x},t)_{\rm rad} = \frac{1}{\mu_0}\mathbf{E}_{\rm md}(\mathbf{x},t)_{\rm rad}\times\mathbf{B}_{\rm md}(\mathbf{x},t)_{\rm rad} = \frac{\mu_0\omega^4 m_0^2}{(4\pi r)^2c^3}\sin^2\theta\cos^2(\omega t_r)\,\hat{\mathbf{r}}.$$

Intensity (average power per unit area transported locally):

$$I_{\rm md} \doteq \langle|\mathbf{S}_{\rm md}|\rangle = \frac{\mu_0\omega^4 m_0^2}{(4\pi r)^2c^3}\sin^2\theta\,\underbrace{\langle\cos^2(\omega t_r)\rangle}_{1/2}.$$

Differential power (transported per solid angle):

$$\frac{dP_{\rm av}^{\rm (md)}}{d\Omega} = r^2\,\hat{\mathbf{r}}\cdot\langle|\mathbf{S}_{\rm md}|\rangle = \frac{\mu_0\omega^4 m_0^2}{32\pi^2c^3}\sin^2\theta = \frac{3P_{\rm av}^{\rm (md)}}{8\pi}\sin^2\theta.$$

Average power (averaged over directions): [lex117]

$$P_{\rm av}^{\rm (md)} \doteq 2\pi\int_0^\pi d\theta\sin\theta\,\frac{dP_{\rm av}^{\rm (md)}}{d\Omega} = \frac{\mu_0\omega^4 m_0^2}{16\pi c^3}\underbrace{\int_0^\pi d\theta\sin^3\theta}_{4/3} = \frac{\mu_0\omega^4 m_0^2}{12\pi c^3}.$$

Electric dipole radiation at arbitrary distance:

Electric dipole is still treated as point source (indicated by subscript as). The distance r between source and field points compared the wavelength λ can be large (indicated by subscript rad), small, or comparable.

- Point-like electric dipole: $\mathbf{p}(t) = p(t)\,\hat{\mathbf{z}}$.
- Asymptotic vector potential: $\mathbf{A}(\mathbf{x},t)_{\rm as} = \dfrac{\mu_0}{4\pi r}\dfrac{d\mathbf{p}}{dt}\bigg|_{t_r}, \quad t_r = t - \dfrac{r}{c}$.
- Conversion to spherical coordinates:

$$\mathbf{A}(\mathbf{x},t)_{\rm as} = \frac{\mu_0}{4\pi r}p'(t_r)\,\hat{\mathbf{z}} = \frac{\mu_0}{4\pi r}p'(t_r)\Big[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\Big].$$

- Asymptotic magnetic field from $\mathbf{B} = \nabla \times \mathbf{A}$: [lex224]

$$\mathbf{B}(\mathbf{x},t)_{\rm as} = \frac{\mu_0}{4\pi}\left[\frac{p''(t_r)}{cr} + \frac{p'(t_r)}{r^2}\right]\sin\theta\,\hat{\boldsymbol{\phi}}.$$

- Asymptotic electric field from Ampère's law, $\dfrac{\partial \mathbf{E}}{\partial t} = c^2\nabla \times \mathbf{B}$: [lex224]

$$\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t)_{\rm as} = \frac{\mu_0 c^2}{4\pi}\left[\cos\theta\left(\frac{2p'(t_r)}{r^3} + \frac{2p''(t_r)}{cr^2}\right)\hat{\mathbf{r}} + \sin\theta\left(\frac{p'(t_r)}{r^3} + \frac{p''(t_r)}{cr^2} + \frac{p'''(t_r)}{c^2 r}\right)\hat{\boldsymbol{\theta}}\right].$$

$$\Rightarrow\ \mathbf{E}(\mathbf{x},t)_{\rm as} = \frac{\mu_0 c^2}{4\pi}\left[\cos\theta\left(\frac{2p(t_r)}{r^3} + \frac{2p'(t_r)}{cr^2}\right)\hat{\mathbf{r}} + \sin\theta\left(\frac{p(t_r)}{r^3} + \frac{p'(t_r)}{cr^2} + \frac{p''(t_r)}{c^2 r}\right)\hat{\boldsymbol{\theta}}\right].$$

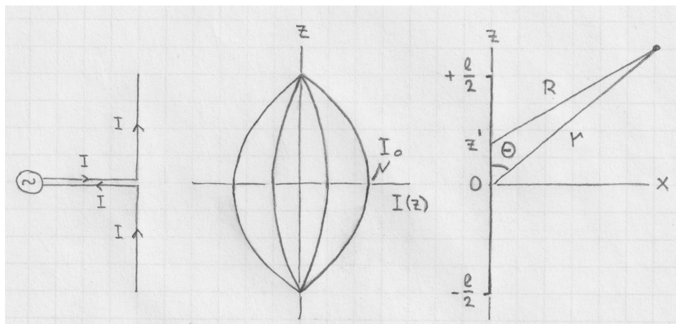
- *Far-field regime*: terms of order p''/r are dominant. Such terms exist in **E** and **B**. Both fields are transverse to radial direction of propagation. This is the radiation zone.
- *Near-field regime*: The dominant term is of order p/r^3, which exist only for **E**. It reflects the instantaneous electric field of an electric dipole. [lex224]
- *Intermediate-field regime*: Here the configuration of electric and magnetic fields is more complex.

Half-wave linear antenna:

Antenna has length ℓ. It is positioned with its center at the origin of the coordinate system and oriented along the z-axis. It carries a (model) current,

$$I(z,t) = I_0 \cos(kz)\sin(\omega t), \quad k = \frac{\pi}{\ell}, \quad \omega = kc = \frac{\pi c}{\ell}.$$

The relation $\lambda \doteq 2\pi/k = 2\ell$ expresses a resonance condition.



Vector potential expression adapted:

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int_{-\ell/2}^{+\ell/2} dz' \, \frac{I\big(z', t - R(z')/c\big)}{R(z')}\,\hat{\mathbf{k}}, \quad R(z') \doteq \sqrt{r^2 - 2rz'\cos\theta + z'^2}.$$

Far-field regime: $R(z') \simeq r - z'\cos\theta, \quad \dfrac{1}{R(z')} \simeq \dfrac{1}{r}.$

Vector potential in radiation zone, using $\ell = \pi/k$:

$$\begin{aligned}
\mathbf{A}(\mathbf{x},t)_{\rm rad} &= \frac{\mu_0}{4\pi r}\int_{-\ell/2}^{+\ell/2} dz' I\left(z', t - \frac{r - z'\cos\theta}{c}\right)\hat{\mathbf{z}} \\
&= \frac{\mu_0 I_0}{4\pi r}\int_{-\ell/2}^{+\ell/2} dz' \cos(kz')\sin(\omega t - kr - kz'\cos\theta)\,\hat{\mathbf{z}} \\
&= -\frac{\mu_0 I_0}{2\pi k r}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\sin(kr - \omega t)\,\hat{\mathbf{z}}.
\end{aligned}$$

Radiation fields and Poynting vector: [lex118]

$$\begin{aligned}
\mathbf{B}(\mathbf{x},t)_{\rm rad} &= \nabla\times\mathbf{A}(\mathbf{x},t)_{\rm rad} = \frac{\mu_0 I_0}{2\pi r}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cos(kr-\omega t)\,\hat{\boldsymbol{\phi}}, \\
\mathbf{E}(\mathbf{x},t)_{\rm rad} &= c\mathbf{B}(\mathbf{x},t)_{\rm rad}\times\hat{\mathbf{r}} = \frac{c\mu_0 I_0}{2\pi r}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cos(kr-\omega t)\,\hat{\boldsymbol{\theta}}, \\
\mathbf{S}(\mathbf{x},t)_{\rm rad} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{c\mu_0 I_0^2}{4\pi^2 r^2}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2\cos^2(kr-\omega t)\,\hat{\mathbf{r}}.
\end{aligned}$$

Radiation from an accelerated charged particle:

A particle with electric charge q has instantaneous position $\mathbf{x}_q(t)$.

- Particle velocity and acceleration: $\mathbf{v} = \dfrac{d\mathbf{x}_q}{dt}, \quad \mathbf{a} = \dfrac{d\mathbf{v}}{dt}$.
- Charge density: $\rho(\mathbf{x}, t) = q\delta\big(\mathbf{x} - \mathbf{x}_q(t)\big)$.
- Current density: $\mathbf{J}(\mathbf{x}, t) = q\mathbf{v}(t)\delta\big(\mathbf{x} - \mathbf{x}_q(t)\big)$.
- Continuity equation, $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$, is satisfied:

$$\nabla \cdot \mathbf{v} = 0 \;\Rightarrow\; \nabla \cdot \mathbf{J} = q\mathbf{v} \cdot \delta\big(\mathbf{x} - \mathbf{x}_q(t)\big), \; \frac{\partial \rho}{\partial t} = q\nabla\delta\big(\mathbf{x} - \mathbf{x}_q(t)\big) \cdot (-\mathbf{v}).$$

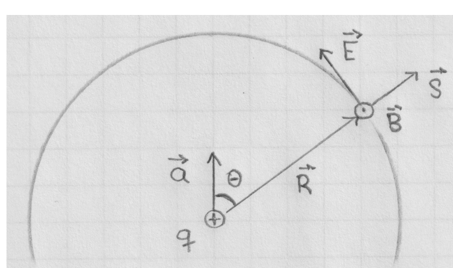
- Electric dipole moment: $\mathbf{p}(t) \doteq \displaystyle\int d^3x\, \mathbf{x}\rho(\mathbf{x}, t) = q\mathbf{x}_q(t)$.
- Distance vector between source and field point: $\mathbf{R}(t_r) \doteq \mathbf{x} - \mathbf{x}_q(t_r)$.
- Retarded time from $t_r = t - |\mathbf{R}(t_r)|/c$.
- Magnetic radiation field from earlier expression for electric dipole:

$$\mathbf{B}(\mathbf{x}, t)_{\text{rad}} = -\frac{\mu_0}{4\pi r c}\hat{\mathbf{R}} \times \left.\frac{d^2\mathbf{p}}{dt^2}\right|_{t_r} = -\frac{\mu_0 q}{4\pi c}\frac{\hat{\mathbf{R}}(t_r) \times \mathbf{a}(t_r)}{R(t_r)}.$$

- Electric radiation field: $\mathbf{E}(\mathbf{x}, t)_{\text{rad}} = c\,\mathbf{B}(\mathbf{x}, t)_{\text{rad}} \times \hat{\mathbf{R}}(t_r)$.
- Poynting vector: [lex119]

$$\mathbf{S}(\mathbf{x}, t)_{\text{rad}} \doteq \frac{1}{\mu_0}\mathbf{E}(\mathbf{x}, t)_{\text{rad}} \times \mathbf{B}(\mathbf{x}, t)_{\text{rad}} = \frac{\mu_0 q^2}{(4\pi)^2 R^2 c}\left[a^2 - \big(\mathbf{a} \cdot \hat{\mathbf{R}}\big)^2\right]\hat{\mathbf{R}}.$$

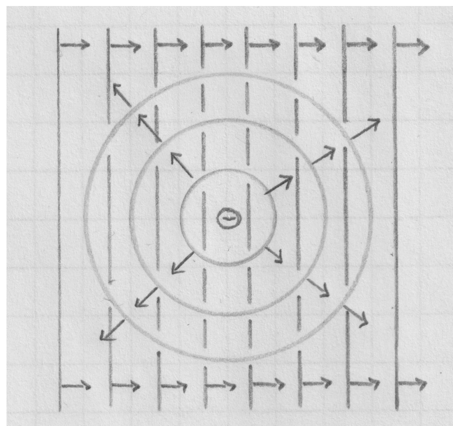
- Instantaneous power radiated: $P(t) = R^2(t_r) \displaystyle\oint d\Omega\, \mathbf{S}(t) \cdot \hat{\mathbf{R}}(t_r)$.
- Larmor formula: $P(t) = \dfrac{1}{4\pi\epsilon_0}\dfrac{2q^2a^2(t_r)}{3c^3}$. [lex119]



Light scattering from bound charged particle:

An incident plane electromagnetic wave agitates a charged particle (e.g. an electron with mass m and charge $-e$) bound to a location in space (e.g. an atom in a crystal).

The resulting accelerated motion of the charged particle generates a radial wave with a specific angular profile (in the radiation zone). Effectively, light is scattered by matter.



Model analyzed in the following: damped harmonic oscillator driven by passing plane electromagnetic wave.

- Oscillating electric field of incident plane wave at position of charged particle: $E_0 e^{-\imath\omega t}\,\hat{\mathbf{i}}$ (dominant driving force).
- Equation of motion: $m\dfrac{d^2x}{dt^2} = -Kx - \gamma\dfrac{dx}{dt} - eE_0 e^{-\imath\omega t}$.
- Natural frequency of oscillation: $\omega_0 \doteq \sqrt{\frac{K}{m}}$ (resonance frequency).
- Steady state solution of this linear ODE (e.g. via Fourier analysis):

$$x(t) = \Re\left[\frac{-eE_0 e^{-\imath\omega t}}{m(\omega_0^2-\omega^2) - \imath\gamma\omega}\right] = \frac{-eE_0 m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\cos(\omega t) + \frac{-eE_0\gamma\omega}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\sin(\omega t).$$

- Acceleration: $a(t) = \dfrac{d^2x}{dt^2}$

$$\Rightarrow\ a(t) = \frac{eE_0\omega^2(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\cos(\omega t) + \frac{eE_0\gamma\omega^3}{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\sin(\omega t).$$

– Average power radiated as predicted by Larmor formula:[1]

$$P_{\rm av} = \langle P(t)\rangle = \frac{e^4 E_0^2 \omega^4}{12\pi\epsilon_0 c^3[m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}.$$

– Classical electron radius:[2] $r_{\rm e} \doteq \dfrac{e^2}{4\pi\epsilon_0 mc^2} \simeq 2.8\times 10^{-15}$m.

[lln15] – Incident beam intensity from earlier: $\langle|\mathbf{S}|\rangle = \dfrac{E_0B_0}{2\mu_0} = \dfrac{1}{2}c\epsilon_0 E_0^2$.

– Scattering cross section defined as average power radiated per incident beam intensity:

$$\sigma \doteq \frac{P_{\rm av}}{\langle|\mathbf{S}|\rangle} = \frac{8\pi r_{\rm e}^2}{3}\frac{\omega^4}{(\omega_0^2-\omega^2)^2+(\gamma\omega/m)^2}.$$

– Rayleigh scattering ($\omega \ll \omega_0$): $\sigma_{\rm Ray} = \dfrac{8\pi r_{\rm e}^2}{3}\left(\dfrac{\omega}{\omega_0}\right)^4$.

– Resonant scattering ($\omega \simeq \omega_0$): sharply peaked for $\gamma/m \ll \omega_0$.

– Thomson scattering($\omega \gg \omega_0$): $\sigma_{\rm Tho} = \dfrac{8\pi r_{\rm e}^2}{3} \simeq 6.6\times 10^{-29}{\rm m}^2$.

Rayleigh scattering explains why the sky is blue and the sunset red. For Thomson scattering the binding and the damping are negligible; the charged particles respond as if they are free.

At yet higher frequencies ($\hbar\omega \simeq mc^2$), quantum effects become critically important. This is the regime of Compton scattering.

[1]For the time averaging we use $\langle\cos^2(\omega t)\rangle = \langle\sin^2(\omega t)\rangle = \frac{1}{2}$.

[2]The classical electron radius is an artificial construct. The electron is assumed to be a sphere of radius $r_{\rm e}$ with charge $-e$ uniformly distributed across its surface. The electrostatic self-energy $e^2/(4\pi\epsilon_0 r_{\rm e})$ is then equated with its rest energy mc^2.