Optics II [lln18]

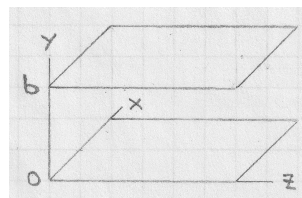
Wave guides and transmission lines operate in the microwave range. Applications include radar, cellphone communication, TV broadcast signals, cavities, among others. Wave guides involve continual interactions between fields and charges. Geometry matters a great deal.

Electromagnetic wave between parallel conducting plates:

Region of space delimited by perfect conductors at $y = 0$ and $y = b$.

Harmonic electromagnetic wave traveling in z-direction:

$$\mathbf{E}(\mathbf{x}, t) = \left[E_x(y)\,\hat{\mathbf{i}} + E_y(y)\,\hat{\mathbf{j}} + E_z(y)\,\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)},$$
$$\mathbf{B}(\mathbf{x}, t) = \left[B_x(y)\,\hat{\mathbf{i}} + B_y(y)\,\hat{\mathbf{j}} + B_z(y)\,\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)}.$$



Maxwell's equations and boundary conditions impose constraints on the six amplitudes in these general expressions.

Harmonic waves have a single wavelength. More general waves are superpositions of harmonic waves.

TEM wave:

In a bounded space, waves that are transverse electric and magnetic (TEM) in the sense that $E_z(y) = B_z(y) \equiv 0$ are an option, not a necessity as is the case in unbounded space (see [lln15]).

A plane-wave solution exists with $\mathbf{E}$ directed perpendicular to the plate, which requires $\mathbf{B}$ to be directed parallel to the plates.

Ansatz: $\mathbf{E}(\mathbf{x}, t) = E_0 \cos(kz - \omega t)\,\hat{\mathbf{j}}, \quad \mathbf{B}(\mathbf{x}, t) = -B_0 \cos(kz - \omega t)\,\hat{\mathbf{i}}.$

Boundary conditions for $\mathbf{E}_\parallel$ and $\mathbf{B}_\perp$ are guaranteed: $\Delta\mathbf{E}_\parallel = 0, \ \Delta\mathbf{B}_\perp = 0.$

Wave travels in direction $\hat{\mathbf{j}} \times (-\hat{\mathbf{i}}) = \hat{\mathbf{k}}$ with speed c.

Gauss's laws, $\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{B} = 0$, are satisfied without further conditions.

Faraday's and Ampère's laws: $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \quad c^2\nabla \times \mathbf{B} = \dfrac{\partial \mathbf{E}}{\partial t}.$

Resulting conditions: $\dfrac{E_0}{B_0} = \dfrac{\omega}{k} = c.$

Poynting vector: $\mathbf{S}(\mathbf{x}, t) \doteq \dfrac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \dfrac{E_0^2}{\mu_0 c}\cos^2(kz - \omega t)\,\hat{\mathbf{k}}.$

Boundary condition for $\mathbf{E}_\perp$ implies surface charges on plates:

Surface charge density [lln6]: $\sigma = \epsilon_0\,\hat{\mathbf{n}}\cdot\mathbf{E}$.

$y = 0: \quad \sigma = \epsilon_0 E_y = \epsilon_0 E_0 \cos(kz - \omega t),$

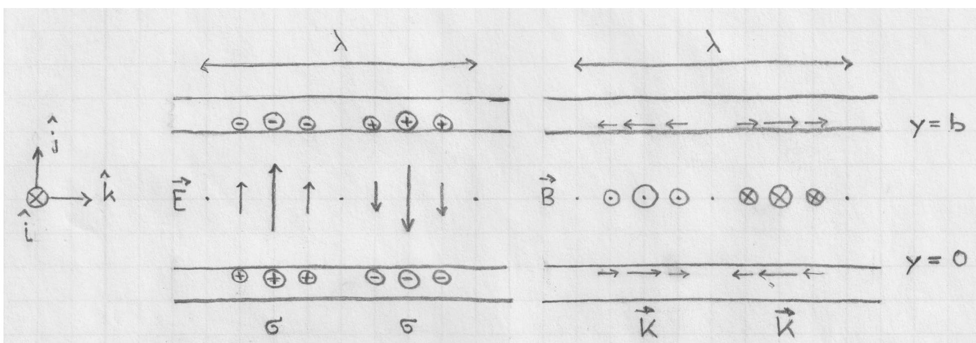
$y = b: \quad \sigma = -\epsilon_0 E_y = -\epsilon_0 E_0 \cos(kz - \omega t).$

Boundary condition for $\mathbf{B}_\parallel$ implies surface currents on plates:

Surface current density [lln13]: $\mathbf{K} = \dfrac{1}{\mu_0}\,\hat{\mathbf{n}}\times\mathbf{B}$.

$y = 0: \quad K_z = -\dfrac{B_x}{\mu_0} = +\dfrac{E_0}{\mu_0 c}\cos(kz - \omega t),$

$y = b: \quad K_z = +\dfrac{B_x}{\mu_0} = -\dfrac{E_0}{\mu_0 c}\cos(kz - \omega t).$



TE wave:

Ansatz: $\mathbf{E}(\mathbf{x},t) = E_x(y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{i}}$ (parallel to the conducting surfaces).

Gauss's law, $\nabla\cdot\mathbf{E} = 0$, is guaranteed.

Faraday's law:[1] $\dfrac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \quad\Rightarrow\quad -\imath\omega\mathbf{B} = -\nabla\times\mathbf{E}$

$$\nabla\times\mathbf{E} = E_x\frac{\partial}{\partial z}\,e^{\imath(kz-\omega t)}\hat{\mathbf{j}} - \frac{\partial E_x}{\partial y}\,e^{\imath(kz-\omega t)}\hat{\mathbf{k}} = \left[\imath k E_x(y)\hat{\mathbf{j}} - E_x'(y)\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)}$$

$$\Rightarrow\ \mathbf{B}(\mathbf{x},t) = \left[\frac{k}{\omega}E_x(y)\,\hat{\mathbf{j}} + \frac{\imath}{\omega}E_x'(y)\,\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)}.$$

Gauss's law, $\nabla\cdot\mathbf{B} = 0$, follows from the identity $\nabla\cdot(\nabla\times\mathbf{E}) = 0$.

[1]The factor $e^{-\imath\omega t}$ must be common to both fields of the same wave.

Ampère's law, $\dfrac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B}$, imposes restrictions on $E_x(y)$ and $\omega(k)$:

$$\nabla \times \mathbf{B} = \left[\frac{\imath}{\omega} \frac{\partial E'_x}{\partial y} e^{\imath(kz-\omega t)} - \frac{k}{\omega} E_x \frac{\partial}{\partial z} e^{\imath(kz-\omega t)} \right] \hat{\mathbf{i}}$$

$$\Rightarrow \quad -\imath\omega E_x(y) = \imath \frac{c^2}{\omega} \left[E''_x(y) - k^2 E_x(y) \right] \quad \Rightarrow \; E''_x(y) = -\left(\frac{\omega^2}{c^2} - k^2 \right) E_x(y).$$

Ansatz for general solution: $E_x(y) = c_1 \sin(\nu y) + c_2 \cos(\nu y)$

$$\Rightarrow \; E''_x(y) = -\nu^2 E_x(y), \quad \frac{\omega^2}{c^2} = k^2 + \nu^2.$$

Boundary conditions for $\mathbf{E}_\parallel$ at surface of perfect conductor: $E_x(0) = E_x(b) = 0$

$$\Rightarrow \; c_2 = 0, \quad \nu b = n\pi \quad \Rightarrow \; E_x(y) = E_0 \sin\left(\frac{n\pi y}{b} \right).$$

Dispersion relation:[2] $\dfrac{\omega^2}{c^2} = k^2 + \left(\dfrac{n\pi}{b} \right)^2, \quad n = 1, 2, \ldots.$

TE(n) mode of TE wave:[3]

$$\mathbf{E}(\mathbf{x}, t) = E_0 \sin\left(\frac{n\pi y}{b} \right) e^{\imath(kz-\omega t)} \, \hat{\mathbf{i}}, \tag{1}$$

$$\mathbf{B}(\mathbf{x}, t) = \left[\frac{k}{\omega} E_0 \sin\left(\frac{n\pi y}{b} \right) \hat{\mathbf{j}} + \imath \frac{n\pi}{b\omega} E_0 \cos\left(\frac{n\pi y}{b} \right) \hat{\mathbf{k}} \right] e^{\imath(kz-\omega t)}. \tag{2}$$

Boundary condition for $\mathbf{B}_\perp$ at the surface of a perfect conductor, $B_y(0) = B_y(b) = 0$, is guaranteed with no further conditions.

The TE wave is translationally invariant in x-direction. It is a traveling wave in z-direction and has standing-wave characteristics in y-direction.

The magnetic field in y-direction is in phase with the electric field (in x-direction). The magnetic field in z-direction is phase shifted by 90°.

Phase velocity and group velocity are different [lex172]:

$$v_{\rm ph} \doteq \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2 - \left(\frac{n\pi c}{b}\right)^2}}, \qquad v_{\rm gr} \doteq \frac{d\omega}{dk} = \frac{c}{\omega} \sqrt{\omega^2 - \left(\frac{n\pi c}{b} \right)^2}.$$

[2] The angular frequency must exceed the threshold value $n\pi/b$.

[3] The physical fields are the real parts of the complex field expressions.

– Phase velocity: phase point in $e^{\imath(kz-\omega t)}$ moves at velocity $\delta z/\delta t = \omega/k$.

– Group velocity: velocity of signals and energy transport.

Boundary condition for $\mathbf{E}_\perp$:

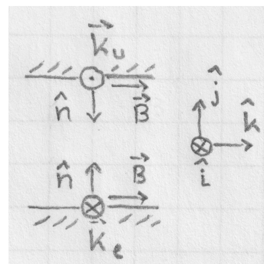
– The solution (1) has $\mathbf{E}_\perp = 0$ at $y = 0$ and $y = b$.

– Implication: zero surface charge density on the surfaces of the perfect conductors at all times ($\sigma \equiv 0$).

Boundary condition for $\mathbf{B}_\parallel$ implies surface currents, $\mathbf{K} = \dfrac{1}{\mu_0}\,\hat{\mathbf{n}} \times \mathbf{B}$:

$$\mathbf{K}_\mathrm{l} = +\frac{B_z(x,0,z,t)}{\mu_0}\,\hat{\mathbf{i}} = -\frac{n\pi E_0}{\mu_0 b\omega}\sin(kz-\omega t)\,\hat{\mathbf{i}},$$

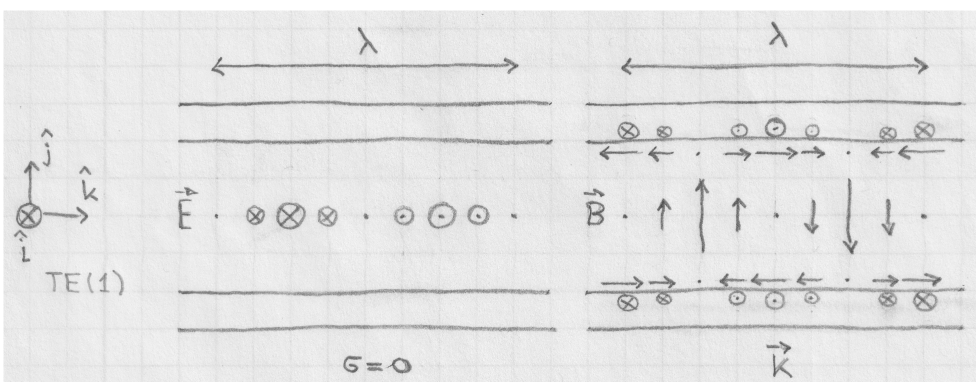
$$\mathbf{K}_\mathrm{u} = -\frac{B_z(x,b,z,t)}{\mu_0}\,\hat{\mathbf{i}} = (-1)^n\frac{n\pi E_0}{\mu_0 b\omega}\sin(kz-\omega t)\,\hat{\mathbf{i}}.$$



Field strengths:

E_x and B_y are strongest in the center,

B_z is strongest near the plates.



Energy transport in TE wave:

Poynting vector: $\mathbf{S} \doteq \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ with fields from (1) and (2)

$$\Rightarrow\ \mathbf{S}(\mathbf{x},t) = \frac{kE_0^2}{\mu_0\omega}\left[\sin^2\left(\frac{n\pi y}{b}\right)\cos^2(kz-\omega t)\,\hat{\mathbf{k}} + \frac{n\pi}{kb}\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t)\cos(kz-\omega t)\,\hat{\mathbf{j}}\right].$$

Energy transport at given position $\mathbf{x}$ has a positive $\hat{\mathbf{k}}$-component and an oscillating $\hat{\mathbf{j}}$-component.

Average power per unit area transported by TE wave:

$$\bar{\mathbf{S}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\,\mathbf{S}(\mathbf{x},t) = \frac{kE_0^2}{2\mu_0\omega}\sin^2\left(\frac{n\pi y}{b}\right)\hat{\mathbf{k}}.$$

Power P per unit (lateral) distance dx carried by TE wave:[4]

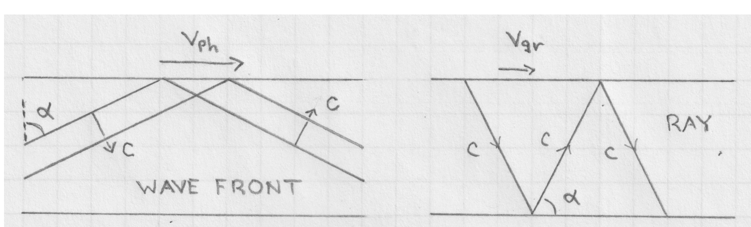
$$\frac{dP}{dx} = \int_0^b dy\,\bar{\mathbf{S}}\cdot\hat{\mathbf{k}} = \frac{E_0^2}{2\mu_0}\frac{k}{\omega}\frac{b}{2} = \frac{bE_0^2}{4\mu_0 c^2}v_{\mathrm{gr}}.$$

Geometrical interpretation of phase/group velocities:

The power is transported at the group velocity v_{gr} in $\hat{\mathbf{k}}$-direction. The wave fronts advance in $\hat{\mathbf{k}}$-direction with the higher phase velocity v_{ph} as the wave fronts zig-zag between the conductors.

The TE wave travels in a zig-zag manner up and down and toward the right. It is periodically reflected at the conducting plates.

At the threshold frequency $\omega \to n\pi c/b$, realized for $\alpha \to \pi/2$, the phase velocity diverges and the group velocity approaches zero: the wave fronts become parallel to the walls.



[4]We use the relation $v_{\mathrm{ph}}v_{\mathrm{gr}} = c^2$ derived in [lex172] for this case.

TM wave:

Ansatz: $\mathbf{B}(\mathbf{x},t) = B_x(y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{i}}$ (parallel to the conducting surfaces).

Gauss's law, $\nabla\cdot\mathbf{B}=0$, is guaranteed.

Boundary condition for $\mathbf{B}_\perp$ at surface of perfect conductor, $B_y(0)=B_y(b)=0$, is guaranteed by construction.

Electric field from Ampère's law: $c^2\nabla\times\mathbf{B} = \dfrac{\partial\mathbf{E}}{\partial t} = -\imath\omega\mathbf{E}$.

$$\nabla\times\mathbf{B} = B_x\frac{\partial}{\partial z}\,e^{\imath(kz-\omega t)}\hat{\mathbf{j}} - \frac{\partial B_x}{\partial y}\,e^{\imath(kz-\omega t)}\hat{\mathbf{k}} = \Big[\imath k B_x(y)\hat{\mathbf{j}} - B_x'(y)\hat{\mathbf{k}}\Big]\,e^{\imath(kz-\omega t)}$$

$$\Rightarrow\ \mathbf{E}(\mathbf{x},t) = -\frac{c^2}{\omega}\Big[kB_x(y)\,\hat{\mathbf{j}} + \imath B_x'(y)\,\hat{\mathbf{k}}\Big]\,e^{\imath(kz-\omega t)}.$$

Gauss's law, $\nabla\cdot\mathbf{E}=0$, follows from the identity $\nabla\cdot(\nabla\times\mathbf{B})=0$.

Wave equation for magnetic field constrains profile of $B_x(y)$:[5]

$$\nabla^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2},\quad\Rightarrow\ B_x''(y) = -\left(\frac{\omega^2}{c^2}-k^2\right)B_x(y).$$

Ansatz for general solution: $B_x(y) = c_1\sin(\nu y) + c_2\cos(\nu y)$

$$\Rightarrow\ B_x''(y) = -\nu^2 B_x(y),\quad \frac{\omega^2}{c^2} = k^2+\nu^2.$$

Boundary conditions for $\mathbf{E}_\parallel$ at surface of perfect conductor, $E_z(0)=E_z(b)=0$.

$$\Rightarrow\ c_1=0,\quad \nu b = n\pi\quad\Rightarrow\ B_x(y) = B_0\cos\left(\frac{n\pi y}{b}\right).$$

Dispersion relation: $\dfrac{\omega^2}{c^2} = k^2 + \left(\dfrac{n\pi}{b}\right)^2,\quad n=1,2,\ldots$ (as for TE wave).

Phase velocity $v_{\rm ph}$ and group velocity $v_{\rm gr}$ are unchanged as well.

TM(n) mode of TM wave:[6]

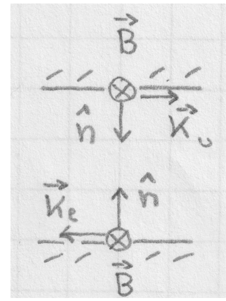
$$\mathbf{B}(\mathbf{x},t) = B_0\cos\left(\frac{n\pi y}{b}\right)e^{\imath(kz-\omega t)}\,\hat{\mathbf{i}}, \tag{3}$$

$$\mathbf{E}(\mathbf{x},t) = -\frac{c^2k}{\omega}B_0\left[\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{j}} - \imath\frac{n\lambda}{2b}\sin\left(\frac{n\pi y}{b}\right)\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)}. \tag{4}$$

[5]This step is equivalent to invoking Faraday's law.

[6]The TEM wave is recovered by setting $n=0$.

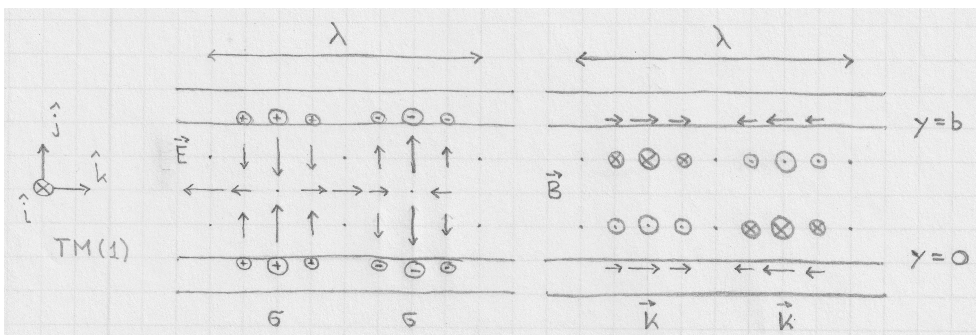
Discontinuity of $\mathbf{B}_{\parallel}$ implies surface currents, $\mathbf{K} = \dfrac{1}{\mu_0}\,\hat{\mathbf{n}} \times \mathbf{B}$:



$$\mathbf{K}_{\mathrm{l}} = -\frac{B_x(x,0,z,t)}{\mu_0}\,\hat{\mathbf{k}} = -\frac{B_0}{\mu_0}\cos(kz-\omega t)\,\hat{\mathbf{k}},$$
$$\mathbf{K}_{\mathrm{u}} = \frac{B_z(x,b,z,t)}{\mu_0}\,\hat{\mathbf{k}} = -(-1)^n\frac{B_0}{\mu_0}\cos(kz-\omega t)\,\hat{\mathbf{k}}.$$

Discontinuity of $\mathbf{E}_{\perp}$ implies surface charges, $\sigma = \epsilon_0\,\hat{\mathbf{n}}\cdot\mathbf{E}$:

$$\sigma_{\mathrm{l}} = \epsilon_0 E_y(x,0,z,t) = -\frac{\epsilon_0 c^2 k}{\omega} B_0\cos(kz-\omega t),$$
$$\sigma_{\mathrm{u}} = -\epsilon_0 E_y(x,b,z,t) = (-1)^n\frac{\epsilon_0 c^2 k}{\omega} B_0\cos(kz-\omega t).$$



Field strengths:

E_z is strongest in the center,

B_x and E_y are strongest near the plates.

Energy transport in TM wave:

Poynting vector: $\mathbf{S} \doteq \dfrac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ from fields (3) and (4):

$$\Rightarrow\ \mathbf{S}(\mathbf{x},t) = \frac{c^2 k B_0^2}{\mu_0\omega}\left[\cos^2\left(\frac{n\pi y}{b}\right)\cos^2(kz-\omega t)\,\hat{\mathbf{k}} - \frac{n\lambda}{2b}\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t)\cos(kz-\omega t)\,\hat{\mathbf{j}}\right].$$

Energy transport has again a non-negative part in positive z-direction and an oscillatory part in y-direction.

In the TE(1) mode, the energy transport was highest at the midpoint between the conductor. In the TM(1) mode it is highest near the conductors.

Average power per unit area transported by TM wave:

$$\bar{\mathbf{S}} \doteq \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, \mathbf{S}(\mathbf{x},t) = \frac{c^2 k B_0^2}{2\mu_0\omega}\cos^2\left(\frac{n\pi y}{b}\right)\hat{\mathbf{k}}.$$

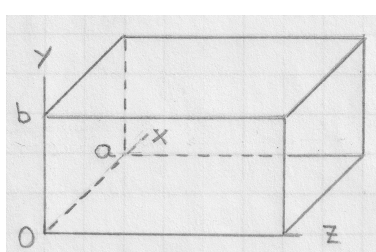
Salient features of TE and TM waves guided by parallel conducting plates:

- The presence of longitudinal fields signals a more complex pattern of wave propagation.
- Waves are characterized as discrete sets of modes, where each mode has a distinct dispersion $\omega(k)$ with a distinct threshold frequency.
- The Poynting vector changes direction periodically, in a zigzag pattern. The wave front is in continual reflection from both walls.
- The zigzag wave propagation slows down the energy transport along the wave guide. The group velocity decreases whereas the phase velocity increases.
- The wave induces surface charges and surface currents on the conductors. This is the cause of attenuation if the walls are less than perfect conductors.

Rectangular wave guide:

The region at $0 < x < a$, $0 < y < b$, $-\infty < z < +\infty$ is surrounded by perfect conductors.

No TEM mode exists in this region.[7] Discrete TE and TM modes do exist. Each such mode is described by two integer quantum numbers n, m.



[7]The general condition for the existene of TEM modes will be discussed later.

TE modes:

General TE expression: $\mathbf{E}(\mathbf{x},t) = \Big[E_x(x,y)\,\hat{\mathbf{i}} + E_y(x,y)\,\hat{\mathbf{j}}\Big]e^{\imath(kz-\omega t)}$.

Ansatz that satisfies Gauss's law, $\nabla\cdot\mathbf{E} = 0$, by construction:

$$\mathbf{E} = \nabla\times\Big[-\psi(x,y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{k}}\Big] = \left[\frac{\partial}{\partial x}\psi(x,y)\,\hat{\mathbf{j}} - \frac{\partial}{\partial y}\psi(x,y)\,\hat{\mathbf{i}}\right]e^{\imath(kz-\omega t)}.$$

The requirement that this expression satisfies the wave equation, can be met by a function $\psi(x,y)$ that satisfies the Helmholtz equation in two dimensions:

$$\nabla^2\mathbf{E} - \frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad \overset{\text{[lex104]}}{\Rightarrow} \quad \nabla^2\psi = -\gamma^2\psi, \quad \gamma^2 = +\frac{\omega^2}{c^2} - k^2.$$

Search for product solutions, $\psi(x,y) = f(x)g(y)$:

$$\Rightarrow \frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} = -\gamma^2 \quad \Rightarrow \frac{d^2f}{dx^2} = -\mu^2 f, \quad \frac{d^2g}{dy^2} = -\nu^2 g, \quad \gamma^2 = \mu^2 + \nu^2.$$

General solution:

$$f(x) = c_1\cos(\mu x) + c_2\sin(\mu x), \quad g(y) = c_3\cos(\nu y) + c_4\sin(\nu y).$$

$$\mathbf{E} = \Big\{\big[-c_1\mu\sin(\mu x) + c_2\mu\cos(\mu x)\big]g(y)\,\hat{\mathbf{j}} + \big[c_3\nu\sin(\nu y) - c_4\nu\cos(\nu y)\big]f(x)\,\hat{\mathbf{i}}\Big\}e^{\imath(kz-\omega t)}.$$

Boundary condition, $\mathbf{E}_\parallel = 0$, at surface of perfect conductor:

$$c_2 = 0, \quad \sin(\mu a) = 0, \qquad c_4 = 0, \quad \sin(\nu b) = 0,$$

$$\Rightarrow \psi(x,y) = \Psi_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

Discrete modes specified by two integer quantum numbers:

$$\mu = \frac{m\pi}{a}, \quad \nu = \frac{n\pi}{b} \quad \Rightarrow \gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \quad m,n = 0,1,2,\ldots.$$

TE(m,n) modes are traveling waves in z-direction and standing waves in both x- and y-directions (with arbitrary amplitude Ψ_0):

$$\mathbf{E}(\mathbf{x},t) = \Psi_0 e^{\imath(kz-\omega t)}\left[\frac{n\pi}{b}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\hat{\mathbf{i}} - \frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{j}}\right].$$

Magnetic field from Faraday's law, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} = \imath\omega\mathbf{B}$ [lex104].

$$\nabla \times \mathbf{E} = -\frac{\partial E_y}{\partial z}\,\hat{\mathbf{i}} + \frac{\partial E_x}{\partial z}\,\hat{\mathbf{j}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\mathbf{k}}.$$

$$\Rightarrow \;\mathbf{B}(\mathbf{x},t) = \frac{k}{\omega}\Psi_0 e^{\imath(kz-\omega t)}\left[\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{i}} + \frac{n\pi}{b}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\hat{\mathbf{j}} + \imath\frac{\gamma^2}{k}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\hat{\mathbf{k}}\right].$$

The electric field is transverse in these modes, but the magnetic field has transverse and longitudinal components. The (longitudinal) z-component is significant at long wavelengths (small k).

Dispersion: $\omega^2 = c^2k^2 + \left(\dfrac{m\pi c}{a}\right)^2 + \left(\dfrac{n\pi c}{b}\right)^2 = c^2k^2 + \omega_{mn}^2.$

TE modes propagate only if $\omega > \omega_{mn} = \sqrt{\left(\dfrac{m\pi c}{a}\right)^2 + \left(\dfrac{n\pi c}{b}\right)^2}.$

Phase velocity: $v_{\mathrm{ph}} \doteq \dfrac{\omega}{k} = \dfrac{c}{\sqrt{1-\omega_{mn}^2/\omega^2}}.$

Group velocity [lex172]: $v_{\mathrm{gr}} \doteq \dfrac{d\omega}{dk} = \dfrac{c^2}{v_{\mathrm{ph}}} = c\sqrt{1-\dfrac{\omega_{mn}^2}{\omega^2}}.$

TM modes:[8]

General TM expression: $\mathbf{B}(\mathbf{x},t) = \Big[B_x(x,y)\,\hat{\mathbf{i}} + B_y(x,y)\,\hat{\mathbf{j}}\Big]e^{\imath(kz-\omega t)}.$

Ansatz that satisfies Gauss's law, $\nabla\cdot\mathbf{B} = 0$, by construction:

$$\mathbf{B} = \nabla \times \Big[-\psi(x,y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{k}}\Big] = \left[\frac{\partial}{\partial x}\psi(x,y)\,\hat{\mathbf{j}} - \frac{\partial}{\partial y}\psi(x,y)\,\hat{\mathbf{i}}\right]e^{\imath(kz-\omega t)}.$$

Boundary conditions, $\mathbf{B}_\perp = 0$, at surface of perfect conductor:

$$\Rightarrow \;\psi(x,y) = \Psi_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

[8]The chain of reasoning varies little from that used for TE modes.

TM(m,n) modes are traveling wave in z-direction and standing wave in both x- and y-directions (with arbitrary amplitude Ψ_0):

$$\mathbf{B}(\mathbf{x},t) = \Psi_0 e^{\imath(kz-\omega t)} \left[-\frac{n\pi}{b} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \hat{\mathbf{i}} + \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \hat{\mathbf{j}} \right].$$

Electric field from Ampère's law, $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = -\frac{\imath\omega}{c^2}\mathbf{E}$.

$$\nabla \times \mathbf{B} = -\frac{\partial B_y}{\partial z}\hat{\mathbf{i}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{j}} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{\mathbf{k}}.$$

$$\Rightarrow \mathbf{E}(\mathbf{x},t) = \frac{c^2 k}{\omega}\Psi_0 e^{\imath(kz-\omega t)} \left[\frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \hat{\mathbf{i}} + \frac{n\pi}{b} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \hat{\mathbf{j}} - \imath\frac{\gamma^2}{k} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \hat{\mathbf{k}} \right].$$

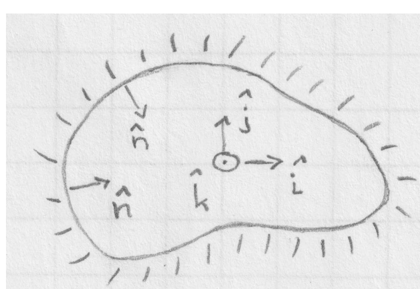
The magnetic field is transverse in these modes, but not the electric field. The longitudinal component is significant at long wavelengths (small k).

The lowest-frequency TM mode is TM(1,1), whereas the lowest-frequency TE mode is either TE(1,0) or TE(0,1).

Wave guide with cross section of arbitrary shape:

Perfect conductor with interior surface specified by cross sectional curve $\mathcal{C}$.

The field expressions in terms of a scalar potential $\psi(x,y)$ remain the same. The boundary conditions stated for $\psi(x,y)$ impose those for the electric and magnetic fields simultaneously (to be explained separately for the TE and TM modes in [lex107] and [lex108]).



TE modes:

Ansatz for electric and magnetic fields [lex107]:

$$\mathbf{E}(\mathbf{x},t) = -\nabla\times\left[\psi(x,y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{k}}\right] = -\left[\nabla\times\psi(x,y)\,\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)},$$
$$\mathbf{B}(\mathbf{x},t) = \frac{k}{\omega}\left[-\nabla\psi(x,y) + \imath\frac{\gamma^2}{k}\psi(x,y)\,\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)}.$$

Maxwell's equations are satisfied [lex104] if $\psi(x,y)$ satisfies the Helmholtz equation,

$$\nabla^2\psi = -\gamma^2\psi, \quad \gamma^2 = \frac{\omega^2}{c^2} - k^2.$$

All relevant boundary conditions are encoded in the condition [lex107],

$$\hat{\mathbf{n}}\cdot\nabla\psi = 0 \quad \text{for all} \;\; \mathbf{x}\in\mathcal{C}.$$

The task has thus been reduced to the solution of an eigenvalue problem with Neumann boundary conditions and a built-in frequency threshold: $\omega \geq c\gamma$.

TM modes:

Ansatz for electric and magnetic fields [lex108]:

$$\mathbf{B}(\mathbf{x},t) = -\nabla\times\left[\psi(x,y)e^{\imath(kz-\omega t)}\,\hat{\mathbf{k}}\right],$$
$$\mathbf{E}(\mathbf{x},t) = \frac{c^2k}{\omega}\left[\nabla\psi(x,y) - \imath\frac{\gamma^2}{k}\psi(x,y)\,\hat{\mathbf{k}}\right]e^{\imath(kz-\omega t)}.$$

Maxwell's equations are satisfied [lex104] if $\psi(x,y)$ satisfies the Helmholtz equation,

$$\nabla^2\psi = -\gamma^2\psi, \quad \gamma^2 = \frac{\omega^2}{c^2} - k^2.$$

All relevant boundary conditions are encoded in [lex108]

$$\psi = 0 \quad \text{for all} \;\; \mathbf{x}\in\mathcal{C}.$$

The task has thus been reduced to the solution of an eigenvalue problem with Dirichlet boundary conditions and a built-in frequency threshold: $\omega \geq c\gamma$.

Conditions for TEM modes:

The conditions for the existence of TEM modes are more restrictive than those for TE or TM modes in a wave guide with a single conducting surface of arbitrary cross section.

- A purely transverse magnetic field in TE modes or a purely transverse electric field in TM modes only exist if $\gamma = 0$.
- In consequence, the Helmholtz equation for any TEM mode reduces to the Laplace equation, $\nabla^2\psi = 0$.
- In regions delimited by a single boundary – one cross-sectional loop C – the trivial solution $\psi(x, y) \equiv 0$ exists and is unique.
- In order to permit TEM modes, the wave guide must consist of at least two different conductors insulated from each other.
- The simplest, most symmetric configuration, apart from parallel plates (analyzed earlier), is a coaxial cable.

TEM mode in coaxial cable:

Two concentric cylindrical surfaces of perfect conductor separated by a region $a < r < b$ of isotropic dielectric material with permittivity ϵ.

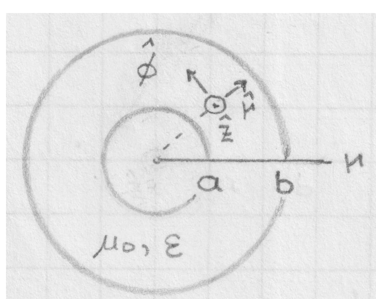
The use of cylindrical coordinates r, ϕ, z is a natural first step.

The boundary conditions at the surface of the conducting surfaces dictate that $\mathbf{E}_\parallel = 0$ and $\mathbf{B}_\perp = 0$.

Cylindrical symmetry dictates that the radial electric field and the azimuthal magnetic field are independent of ϕ and z.

Ansatz for electric and magnetic fields of TEM mode:

$$\mathbf{E}(\mathbf{x}, t) = E_r(r)e^{\imath(kz-\omega t)}\,\hat{\mathbf{r}}, \quad \mathbf{B}(\mathbf{x}, t) = B_\phi(r)e^{\imath(kz-\omega t)}\,\hat{\boldsymbol{\phi}}.$$



The functions $E_r(r)$ and $B_\phi(r)$ and the dispersion $\omega(k)$ follow from substitution of the ansatz into Maxwell's equations [lex109]:

$$\mathbf{E}(\mathbf{x},t) = \frac{C_0}{r}\, e^{\imath(kz-\omega t)}\,\hat{\mathbf{r}}, \quad \mathbf{B}(\mathbf{x},t) = \frac{C_0}{vr}\, e^{\imath(kz-\omega t)}\,\hat{\boldsymbol{\phi}},$$

where

$$\frac{\omega}{k} = v = \frac{1}{\sqrt{\mu_0 \epsilon}}$$

is the (dispersionless) wave velocity. The amplitude of the electric potential difference between the conductors is

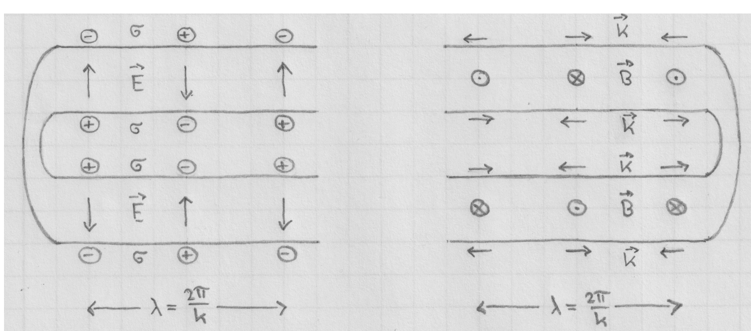
$$\Phi_0 = C_0 \ln(b/a).$$

The two remaining boundary conditions, for $\mathbf{E}_\perp$ and $\mathbf{B}_\parallel$, determine the surface charge density,

$$\sigma = \epsilon E_n = \begin{cases} \dfrac{+\epsilon C_0}{a}\, e^{\imath(kz-\omega t)} & : \ r=a, \\ \dfrac{-\epsilon C_0}{b}\, e^{\imath(kz-\omega t)} & : \ r=b, \end{cases}$$

and the surface current densities,

$$\mathbf{K} = \frac{1}{\mu_0}\hat{\mathbf{n}} \times \mathbf{B}_\parallel = \begin{cases} \dfrac{+v\epsilon C_0}{a}\, e^{\imath(kz-\omega t)}\,\hat{\mathbf{z}} & : \ r=a, \\ \dfrac{-v\epsilon C_0}{b}\, e^{\imath(kz-\omega t)}\,\hat{\mathbf{z}} & : \ r=b. \end{cases}$$



Exercises:

- ▷ TE mode in rectangular wave guide [lex104]
- ▷ Surface charge and current in rectangular wave guide I [lex105]
- ▷ Surface charge and current in rectangular wave guide II [lex106]
- ▷ Helmholtz equation for wave guide I: TE modes [lex107]
- ▷ Helmholtz equation for wave guide II: TM modes [lex108]
- ▷ TEM mode in coaxial cable I: electric and magnetic fields [lex109]
- ▷ TEM mode in coaxial cable II: impedance [lex110]