Optics I [lln17]

Optics is the study of light and its interaction with conductors and dielectrics. The strongest interaction is between electric field (of light) and electric charges (of electrons in matter).

Electromagnetic wave in dielectric:

Dielectrics have permittivity ϵ, permeability μ, and conductivity $\sigma = 0$.

Constitutive equations: $\mathbf{D}(\mathbf{x},t) = \epsilon\mathbf{E}(\mathbf{x},t), \quad \mathbf{H}(\mathbf{x},t) = \mathbf{B}(\mathbf{x},t)/\mu$.

Maxwell's equations in uniform dielectric (with $\rho_\mathrm{f} = 0$, $\mathbf{J}_\mathrm{f} = 0$):

$$\nabla\cdot\mathbf{D} = 0, \quad \nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}.$$

Wave speed in dielectric: $v = \dfrac{1}{\sqrt{\epsilon\mu}} = \dfrac{c}{n}$.

Index of refraction: $n = \dfrac{c}{v} = \sqrt{\dfrac{\epsilon\mu}{\epsilon_0\mu_0}}$ (in general, frequency-dependent).

Linearly polarized plane electromagnetic wave:

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \quad \mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)}.$$

Transverse nature of wave, $\mathbf{E}\cdot\mathbf{k} = \mathbf{B}\cdot\mathbf{k} = 0$, follows from Gauss's laws:

$$\nabla\cdot(\epsilon\mathbf{E}) = \epsilon\imath\mathbf{k}\cdot\mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0, \quad \nabla\cdot\mathbf{B} = \imath\mathbf{k}\cdot\mathbf{B}_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)} = 0.$$

Faraday's law and Ampère's law establish (i) the right-handed orthogonal triad between vectors $\mathbf{E}_0, \mathbf{B}_0, \mathbf{k}$, and (ii) the ratio between amplitudes $\mathbf{E}_0, \mathbf{B}_0$:

$$\imath\mathbf{k}\times\mathbf{E} = \imath\omega\mathbf{B}, \quad \imath\mathbf{k}\times\frac{\mathbf{B}}{\mu} = -\imath\omega\epsilon\mathbf{E}.$$

$$\Rightarrow \text{ (i) } \frac{\mathbf{E}_0}{E_0}\times\frac{\mathbf{B}_0}{B_0} = \frac{\mathbf{k}}{k}, \qquad \text{(ii) } \frac{E_0}{B_0} = \frac{\omega}{k} = v.$$

Distinction between phase velocity and group velocity:

$$v_\mathrm{phase} \doteq \frac{\omega}{k}, \quad v_\mathrm{group} \doteq \frac{d\omega}{dk}. \tag{1}$$

Dispersion relation: function $\omega(k)$. If $\omega = ck$ then $v_\mathrm{phase} = v_\mathrm{group} = c$.

Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H} = \dfrac{1}{\mu}\mathbf{E} \times \mathbf{B}$.

$$\Rightarrow \ \mathbf{S}(\mathbf{x},t) = \frac{1}{\mu}\mathbf{E}_0 \times \mathbf{B}_0 \cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t) = \frac{E_0^2}{\mu v}\frac{\mathbf{k}}{k}\cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t).$$

Intensity: $I = \left\langle \dfrac{\mathbf{k}}{k}\cdot\mathbf{S} \right\rangle = \dfrac{1}{\mu v}\,E_0^2\,\underbrace{\langle\cos^2(\mathbf{k}\cdot\mathbf{x} - \omega t)\rangle}_{1/2} = \dfrac{1}{2}\epsilon v E_0^2.$

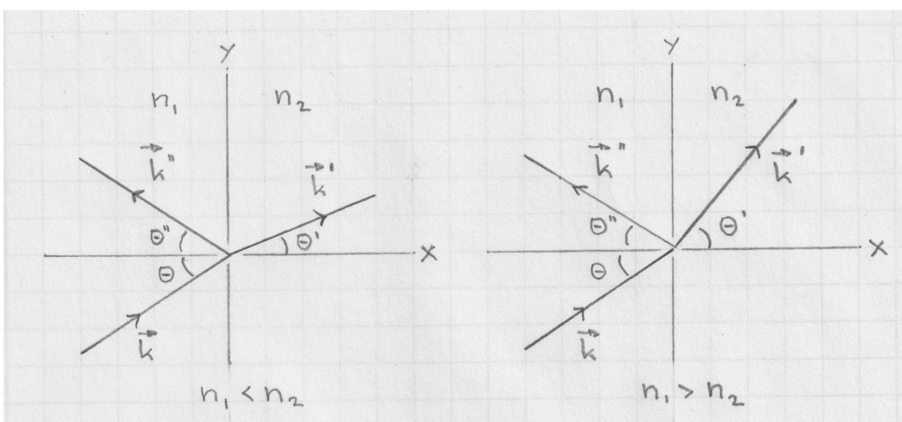
Reflection and refraction at plane dielectric interface:

Reflection and refraction are consequences of the change in phase velocity and the boundary conditions at the interface as established earlier.

Dielectric media have indices of refraction n_1 and n_2.

Incident, reflected, and refracted plane wave are characterized by rays with wave vectors, $\mathbf{k}$, $\mathbf{k}''$, and $\mathbf{k}'$, respectively.

Reflection and refraction take place in *plane of incidence*, defined by $\mathbf{k}$ and the normal to the interface (here the xy-plane).



Incident, reflected, and refracted waves fully characterized by electric field:

$$\mathbf{E}(\mathbf{x},t) = \begin{cases} \mathbf{E}_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \mathbf{E}_0'' e^{\imath(\mathbf{k}''\cdot\mathbf{x}-\omega t)} & : \ x < 0, \\ \mathbf{E}_0' e^{\imath(\mathbf{k}'\cdot\mathbf{x}-\omega t)} & : \ x > 0. \end{cases}$$

Boundary conditions for dielectric interface (from [lln15]):

$$\begin{aligned} \epsilon_1\mathbf{E}_{1\perp} &= \epsilon_2\mathbf{E}_{2\perp}, \qquad & \mathbf{B}_{1\perp} &= \mathbf{B}_{2\perp}, \\ \mathbf{E}_{1\parallel} &= \mathbf{E}_{2\parallel}, \qquad & \mathbf{B}_{1\parallel}/\mu_1 &= \mathbf{B}_{2\parallel}/\mu_2. \end{aligned}$$

Consequences of symmetry and changing wave speed for angular frequencies and wave vectors:

$$[1]: \ \omega = \omega' = \omega''; \qquad [2]: \ k'' = k, \quad \frac{k'}{n_2} = \frac{k}{n_1}; \qquad [3]: \ k_y = k'_y = k''_y.$$

When the phase velocity ω/k changes between media, it is the wavelength $\lambda = 2\pi/k$ that changes, not the period $T = 2\pi/\omega$. The last consequence follows from translational symmetry along the interface.

Representation of wave vectors satisfying consequences [1] and [2]:

$$\begin{aligned}
\mathbf{k} &= n_1 \frac{\omega}{c}\big[\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\big], \\
\mathbf{k}' &= n_2 \frac{\omega}{c}\big[\cos\theta'\,\hat{\mathbf{i}} + \sin\theta'\,\hat{\mathbf{j}}\big], \\
\mathbf{k}'' &= n_1 \frac{\omega}{c}\big[-\cos\theta''\,\hat{\mathbf{i}} + \sin\theta''\,\hat{\mathbf{j}}\big].
\end{aligned}$$

Implication of consequence [3]: law of reflection and Snell's law of refraction.

$$\sin\theta = \sin\theta'', \qquad n_1 \sin\theta = n_2 \sin\theta'. \tag{2}$$

We save the question of how the amplitudes $\mathbf{E}_0$, $\mathbf{E}'_0$, and $\mathbf{E}''_0$ are related to each other for later.

Total internal reflection:

The index of refraction n is a measure of optical density.

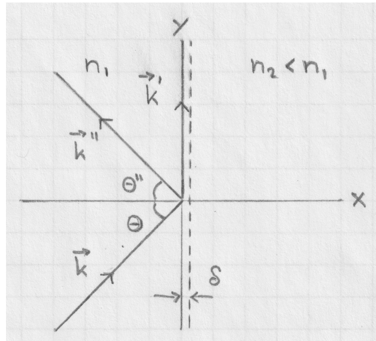
If $n_1 > n_2$, refraction turns into total internal reflection if the angle of incidence exceeds the critical angle,

$$\underbrace{\sin\theta}_{\to\sin\theta_\mathrm{c}} = \frac{n_2}{n_1}\underbrace{\sin\theta'}_{\to 1} \quad \Rightarrow \quad \sin\theta_\mathrm{c} = \frac{n_2}{n_1}.$$

For $\theta > \theta_\mathrm{c}$ we can still formally write,

$$\sin^2\theta = \sin^2\theta_\mathrm{c}\sin^2\theta' \quad \Rightarrow \quad \sin^2\theta' = \frac{\sin^2\theta}{\sin^2\theta_\mathrm{c}} > 1.$$

The refracted wave for $\theta > \theta_\mathrm{c}$ travels along interface and is exponentially attenuated in the optically less dense medium. It is an evanescent wave there and can only be detected in a thin layer close to the interface.



Transformation of the expression for the refracted wave in two steps:

$$\begin{aligned}
\mathbf{E}'(\mathbf{x},t) &= \mathbf{E}'_0 \exp\left(\imath n_2 \frac{\omega}{c} x \cos\theta' + \imath n_2 \frac{\omega}{c} y \sin\theta' - \omega t\right) \\
&= \mathbf{E}'_0 \exp\left(\imath n_1 \frac{\omega}{c} y \sin\theta - \omega t\right) \exp\left(n_2 \frac{\omega}{c} \imath \cos\theta'\right) \\
&= \mathbf{E}'_0 \exp\left(\imath n_1 \frac{\omega}{c} y \sin\theta - \omega t\right) \underbrace{\exp\left(-n_2 \frac{\omega}{c} \xi x\right)}_{e^{-x/\delta}},
\end{aligned}$$

– In the first step we use: $n_1 \sin\theta = n_2 \sin\theta'$

– In the second step we use:

$$\imath \cos\theta' = \imath\sqrt{1-\sin^2\theta'} = -\sqrt{\sin^2\theta' - 1} = -\sqrt{\frac{\sin^2\theta}{\sin^2\theta_\mathrm{c}} - 1} \doteq -\xi,$$

where $\delta \doteq c/(n_2 \omega \xi)$ is the (θ-dependent) attenuation length.

Normal-incidence reflectivity:

Reflectivity and transmissivity are defined as intensity ratios:

$$R \doteq \frac{I''}{I}, \qquad T \doteq \frac{I'}{I}.$$

Normal incidence: $\theta = \theta' = \theta'' = 0$.

Linearly polarized waves with $\mathbf{k} = k\,\hat{\mathbf{i}}$, $\mathbf{k}' = k'\,\hat{\mathbf{i}}$, $\mathbf{k}'' = -k''\,\hat{\mathbf{i}}$, $k'' = k$.

Electric-field polarizations and amplitudes:

$$\mathbf{E}_0 = E_0\,\hat{\mathbf{j}}, \quad \mathbf{E}'_0 = E'_0\,\hat{\mathbf{j}}, \quad \mathbf{E}''_0 = E''_0\,\hat{\mathbf{j}}.$$

Magnetic-field polarizations and amplitudes:[1]

$$\mathbf{H}_0 = \frac{E_0}{\mu_1 v_1}\,\hat{\mathbf{k}}, \quad \mathbf{H}'_0 = \frac{E'_0}{\mu_2 v_2}\,\hat{\mathbf{k}}, \quad \mathbf{H}''_0 = -\frac{E''_0}{\mu_1 v_1}\,\hat{\mathbf{k}}.$$

[1]Distinguish between $\mathbf{k}$ (wave vector) and $\hat{\mathbf{k}}$ (unit vector in z-direction).

Implementation of boundary conditions for tangential field components:

$$E_0 + E_0'' = E_0', \qquad \frac{E_0 - E_0''}{\mu_1 v_1} = \frac{E_0'}{\mu_2 v_2}; \qquad v_1 = \frac{c}{n_1}, \quad v_2 = \frac{c}{n_2}.$$

Electric-field phase relations between waves:

$$\frac{E_0'}{E_0} = \frac{2\mu_2 n_1}{\mu_2 n_1 + \mu_1 n_2}, \qquad \frac{E_0''}{E_0} = \frac{\mu_2 n_1 - \mu_1 n_2}{\mu_2 n_1 + \mu_1 n_2}.$$

▷ $E_0'/E_0 > 0$ is always realized, implying that the incident and refractive waves are always in phase.

The direction of $\mathbf{E}_0$ and $\mathbf{E}_0'$ as well as directions of $\mathbf{H}_0$ and $\mathbf{H}_0'$ are always the same.

▷ E_0''/E_0 can be positive or negative, implying that the reflected wave is in phase with the incident wave if $\mu_2 n_1 > \mu_1 n_2$ and has opposite phase if $\mu_2 n_1 < \mu_1 n_2$.

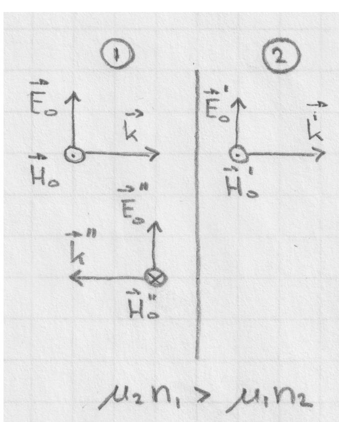
In phase means that $\mathbf{E}_0$ and $\mathbf{E}_0''$ have the same direction. Opposite phase means that $\mathbf{H}_0$ and $\mathbf{H}_0''$ have the same direction.

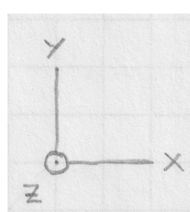
In either case, $\mathbf{E_0} \times \mathbf{H}_0$ and $\mathbf{E_0}'' \times \mathbf{H}_0''$ have opposite direction.

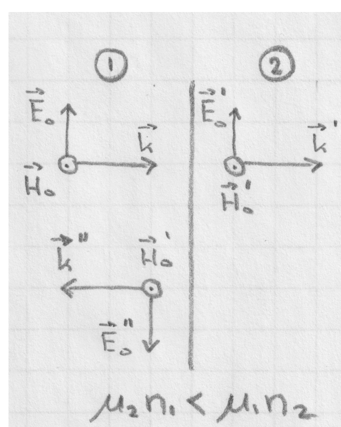
Reflectivity and transmittivity:

$$R = \frac{E_0''^2}{E_0^2} = \left(\frac{\mu_2 n_1 - \mu_1 n_2}{\mu_2 n_1 + \mu_1 n_2}\right)^2, \qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\frac{E_0'^2}{E_0^2} = \frac{\mu_1 n_2}{\mu_2 n_1}\frac{E_0'^2}{E_0^2} = \frac{4\mu_1\mu_2 n_1 n_2}{(\mu_2 n_1 + \mu_1 n_2)^2}.$$

Energy conservation is encoded in the relation, $R + T = 1$.

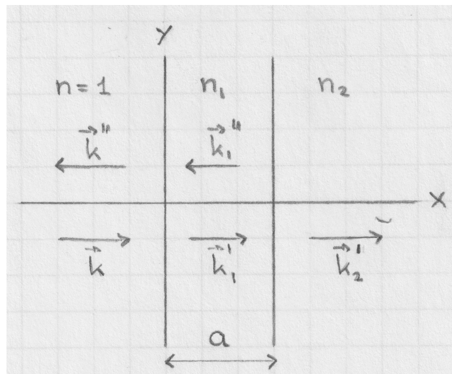






Nonreflecting surface via dielectric coating:

Surface in yz-plane. Dielectric with index n_2 at $x > a$. Surface coating with index n_1 at $0 < x < a$. Air at $x < 0$.



Incident (linearly polarized) plane wave from left (air) is refracted and reflected both at the surface ($x = 0$) and at the interface ($x = a$).

Simplified notation with $k = \omega/c$, $k_1 = n_1 k$, $k_2 = n_2 k$ and $\mathbf{k}\cdot\mathbf{x} = kx$, $\mathbf{k}''\cdot\mathbf{x} = -kx$, $\mathbf{k}'_1\cdot\mathbf{x} = k_1 x$, $\mathbf{k}''_1\cdot\mathbf{x} = -k_1 x$, $\mathbf{k}'_2\cdot\mathbf{x} = k_2 x$:

$$\mathbf{E}(\mathbf{x},t) = \hat{\mathbf{j}} \begin{cases} E_0 e^{\imath(kx-\omega t)} + E''_0 e^{-\imath(kx+\omega t)} & : \ x \leq 0, \\ E'_1 e^{\imath(k_1 x-\omega t)} + E''_1 e^{-\imath(k_1 x+\omega t)} & : \ 0 \leq x \leq a, \\ E'_2 e^{\imath(k_2 x-\omega t)} & : \ x \geq a, \end{cases}$$

$$c\mathbf{B}(\mathbf{x},t) = \hat{\mathbf{k}} \begin{cases} E_0 e^{\imath(kx-\omega t)} - E''_0 e^{-\imath(kx+\omega t)} & : \ x \leq 0, \\ n_1 E'_1 e^{\imath(k_1 x-\omega t)} - n_1 E''_1 e^{-\imath(k_1 x+\omega t)} & : \ 0 \leq x \leq a, \\ n_2 E'_2 e^{\imath(k_2 x-\omega t)} & : \ x \geq a. \end{cases}$$

Boundary conditions for tangential fields at $x = 0$:[2]

$$E_0 + E''_0 = E'_1 + E''_1, \quad E_0 - E''_0 = n_1(E'_1 - E''_1).$$

Boundary conditions for tangential fields at $x = a$:

$$E'_1 e^{\imath k_1 a} + E''_1 e^{-\imath k_1 a} = E'_2 e^{\imath k_2 a}, \quad n_1\big(E'_1 e^{\imath k_1 a} - E''_1 e^{-\imath k_1 a}\big) = n_2 E'_2 e^{\imath k_2 a}.$$

The four boundary conditions determine the values of E''_0, E'_1, E''_1, E'_2 for given value of E_0 of the incident wave, indices of refraction n_1, n_2, and thickness a of the coating.

Condition of zero reflectivity at surface ($x = 0$): $E''_0 = 0$.

[2]We set $\mu_1 = \mu_2 = \mu_0$ and use $H_0 = E_0/\mu_0 c$, $H''_0 = -E''_0/\mu_0 c$, $H'_1 = n_1 E'_1/\mu_0 c$, $H''_1 = -n_1 E'_1/\mu_0 c$, $H'_2 = n_2 E'_2/\mu_0 c$. We also assume that $0 < n_1 < n_2$.

Use boundary conditions at $x = 0$ with $E_0'' = 0$ [lex97]:

$$\Rightarrow \ E_1' = \frac{E_0}{2}\left(1 + \frac{1}{n_1}\right), \quad E_1'' = \frac{E_0}{2}\left(1 - \frac{1}{n_1}\right).$$

Use boundary conditions at $x = a$ [lex97]:

$$\Rightarrow \ \frac{n_2}{n_1} = \frac{\imath\, n_1 \sin(k_1 a) + \cos(k_1 a)}{n_1 \cos(k_1 a) + \imath \sin(k_1 a)}.$$

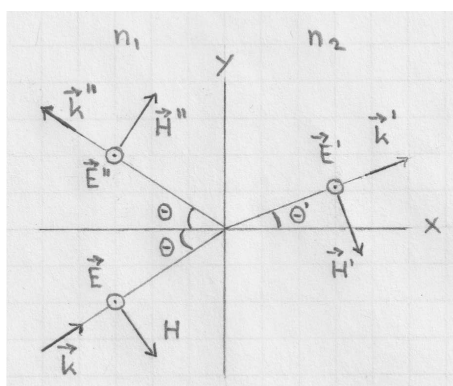
Physically relevant solution [lex97]: $\cos(k_1 a) = 0 \quad \Rightarrow \ a = \dfrac{\lambda_1}{4}, \quad n_1 = \sqrt{n_2}.$

Plane-wave incidence at any angle:

Distinction between two kinds of linear polarizations:

- TE polarization: electric field is perpendicular to plane of incidence.
- TM polarization: magnetic field is perpendicular to plane of incidence.

Reflection and refraction of TE plane wave:



Electric field and magnetic field:

$$\mathbf{E} = E_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,\hat{\mathbf{k}}, \qquad \mathbf{H} = \frac{E_0}{\mu_1 v_1} e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)}\left[\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}\right],$$

$$\mathbf{E}' = E_0' e^{\imath(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\,\hat{\mathbf{k}}, \qquad \mathbf{H}' = \frac{E_0'}{\mu_2 v_2} e^{\imath(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\left[\sin\theta'\,\hat{\mathbf{i}} - \cos\theta'\,\hat{\mathbf{j}}\right],$$

$$\mathbf{E}'' = E_0'' e^{\imath(\mathbf{k}''\cdot\mathbf{x}-\omega t)}\,\hat{\mathbf{k}}, \qquad \mathbf{H}'' = \frac{E_0''}{\mu_1 v_1} e^{\imath(\mathbf{k}''\cdot\mathbf{x}-\omega t)}\left[\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right].$$

Boundary conditions (use $v_1 = c/n_1$, $v_2 = c/n_2$):

$\mathbf{D}_{1\perp} = \mathbf{D}_{2\perp}$: guaranteed by polarization (both vanish).

$\mathbf{E}_{1\|} = \mathbf{E}_{2\|}$: $E_0 + E_0'' = E_0'$.

$\mathbf{B}_{1\perp} = \mathbf{B}_{2\perp}$: equivalent to $\mathbf{E}_{1\|} = \mathbf{E}_{2\|}$ by Snell's law.[3]

$\mathbf{H}_{1\|} = \mathbf{H}_{2\|}$: $\dfrac{n_1}{\mu_1}(E_0 - E_0'')\cos\theta = \dfrac{n_2}{\mu_2}E_0'\cos\theta'$.

Resulting amplitude ratios [lex98]:

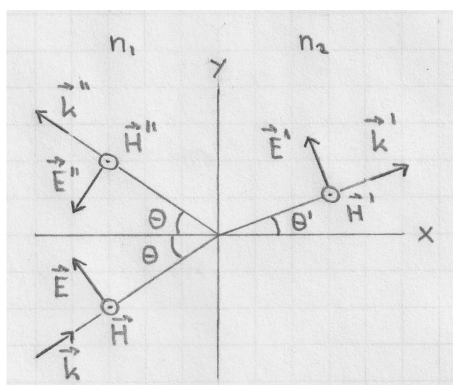
$$\frac{E_0'}{E_0} = \frac{2\mu_2 n_1 \cos\theta}{\mu_2 n_1\cos\theta + \mu_1 n_2 \cos\theta'}, \quad \frac{E_0''}{E_0} = \frac{\mu_2 n_1\cos\theta - \mu_1 n_2\cos\theta'}{\mu_2 n_1\cos\theta + \mu_1 n_2\cos\theta'}.$$

Fresnel's equation: assume $\mu_1 = \mu_2 = \mu_0$ and use Snell's law [lex98].

$$\Rightarrow \frac{E_0''}{E_0} = \frac{\sin(\theta' - \theta)}{\sin(\theta' + \theta)}.$$

Note: $E_0''/E_0 < 0$ indicates a 180° phase change of the reflected wave.[4]

Reflection and refraction of TM plane wave:



Magnetic field and electric field:

$$\mathbf{H} = \frac{E_0}{\mu_1 v_1} e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,\hat{\mathbf{k}}, \qquad \mathbf{E} = E_0 e^{\imath(\mathbf{k}\cdot\mathbf{x}-\omega t)}\Big[-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\Big],$$

$$\mathbf{H}' = \frac{E_0'}{\mu_2 v_2} e^{\imath(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\,\hat{\mathbf{k}}, \qquad \mathbf{E}' = E_0' e^{\imath(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\Big[-\sin\theta'\,\hat{\mathbf{i}} + \cos\theta'\,\hat{\mathbf{j}}\Big],$$

$$\mathbf{H}'' = \frac{E_0''}{\mu_1 v_1} e^{\imath(\mathbf{k}''\cdot\mathbf{x}-\omega t)}\,\hat{\mathbf{k}}, \qquad \mathbf{E}'' = E_0'' e^{\imath(\mathbf{k}''\cdot\mathbf{x}-\omega t)}\Big[-\sin\theta\,\hat{\mathbf{i}} - \cos\theta\,\hat{\mathbf{j}}\Big].$$

[3] $(E_0 + E_0'')n_1\sin\theta = E_0' n_2 \sin\theta'$, $\quad n_1\sin\theta = n_2\sin\theta'$.

[4] The graph is for $E_0''/E_0 > 0$.

Boundary conditions (use $\epsilon_1 = n_1/cv_1\mu_1$, $\epsilon_2 = n_2/cv_2\mu_2$):

$\mathbf{B}_{1\perp} = \mathbf{B}_{2\perp}$: guaranteed by polarization (both vanish).

$\mathbf{H}_{1\|} = \mathbf{H}_{2\|}$: $\dfrac{n_1}{\mu_1}(E_0 + E_0'') = \dfrac{n_2}{\mu_2}E_0'$.

$\mathbf{D}_{1\perp} = \mathbf{D}_{2\perp}$: equivalent to $\mathbf{H}_{1\|} = \mathbf{H}_{2\|}$ by Snell's law.[5]

$\mathbf{E}_{1\|} = \mathbf{E}_{2\|}$: $(E_0 - E_0'')\cos\theta = E_0'\cos\theta'$.

Resulting amplitude ratios [lex99]:

$$\frac{E_0'}{E_0} = \frac{2\mu_2 n_1 \cos\theta}{\mu_1 n_2 \cos\theta + \mu_2 n_1 \cos\theta'}, \quad \frac{E_0''}{E_0} = \frac{\mu_1 n_2 \cos\theta - \mu_2 n_1 \cos\theta'}{\mu_1 n_2 \cos\theta + \mu_2 n_1 \cos\theta'}.$$

Fresnel's equation: assume $\mu_1 = \mu_2 = \mu_0$ and use Snell's law [lex99].

$$\Rightarrow \frac{E_0''}{E_0} = \frac{\tan(\theta - \theta')}{\tan(\theta + \theta')}.$$

Note the different sign conventions for TE and TM waves: here $E_0''/E_0 > 0$ indicates 180° phase change of reflected wave.[6]

Brewster angle:

The TM wave has zero reflectivity for a particular angle of the incident wave.

Brewster angle: $\theta_{\mathrm{B}} = \arctan\left(\dfrac{n_2}{n_1}\right)$ (assuming $\mu_1 = \mu_2 = \mu_0$).

- Condition of zero reflectivity: $\dfrac{E_0''}{E_0} = 0 \quad \Rightarrow \quad \tan(\theta + \theta') = \infty$

  $\Rightarrow \quad \theta + \theta' = \dfrac{\pi}{2} \quad \Rightarrow \quad \theta' = \dfrac{\pi}{2} - \theta.$
- Snell's law: $n_1 \sin\theta = n_2 \sin\theta'$,
- Brewster angle: $n_1 \sin\theta_{\mathrm{B}} = n_2 \sin\theta_{\mathrm{B}}' = n_2 \sin\left(\dfrac{\pi}{2} - \theta_{\mathrm{B}}\right) = n_2 \cos\theta_{\mathrm{B}}$,

  $\Rightarrow \quad \tan\theta_{\mathrm{B}} = \dfrac{n_2}{n_1}.$

Consequence: unpolarized incident light reflected from a dielectric at the Brewster angle θ_{B} becomes fully polarized.

[5] $-(E_0 + E_0'')\epsilon_1 \sin\theta = -E_0'\epsilon_2 \sin\theta'$, $n_1 \sin\theta = n_2 \sin\theta'$.

[6] The graph is for the case $E_0''/E_0 < 0$.

Energy conservation:

Reflectivity and transmittivity have earlier been defined as intensity ratios:[7]

$$R \doteq \frac{I''}{I} = \frac{E_0''^2}{E_0^2}, \quad T \doteq \frac{I'}{I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\frac{E_0'^2}{E_0^2} = \frac{\mu_1 n_2}{\mu_2 n_1}\frac{E_0'^2}{E_0^2}.$$

For normal incidence, we found that $R+T=1$. The intensity of the incident wave is split into the intensities of the reflected and transmitted waves. Total intensity is conserved.

For incidence at an angle, energy conservation does no longer imply conservation of total intensity.

Intensity is power per unit cross sectional area of the wave. Energy conservation applies to power per area of the interface:

$$I\cos\theta = I''\cos\theta + I'\cos\theta' \quad \Rightarrow \; R + T\,\frac{\cos\theta'}{\cos\theta} = 1.$$

Electromagnetic waves in a conductor:

Here the dominant interaction is with (free) conduction electrons.

Density of free current: $\mathbf{J}_f(\mathbf{x},t) = \sigma\mathbf{E}(\mathbf{x},t)$ (Ohm's law in operation).

Linear field relations: $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$.

Maxwell's equations in a metal:

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial\mathbf{E}}{\partial t}.$$

The first term in Ampère's law involves energy dissipation.

Separating the fields by introducing second derivatives and identities of vector analysis, yields identical wave equations for the electric and magnetic fields amended by an attenuation term [lex100]:

$$\nabla^2\mathbf{E} = \mu\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2}, \qquad \nabla^2\mathbf{B} = \mu\sigma\frac{\partial\mathbf{B}}{\partial t} + \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2}.$$

Ansatz for a transverse plane wave traveling in x-direction:

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{\imath(\kappa x - \omega t)}, \quad \mathbf{E}_0 \perp \hat{\mathbf{i}}.$$

[7] In the last equality we have used $v_1 = c/n_1 = c/\sqrt{\epsilon_1\mu_1}$ and $v_2 = c/n_2 = c/\sqrt{\epsilon_2\mu_2}$.

Substitution of the ansatz into the amended wave equation requires that the parameters κ and ω satisfy the relation [lex100],

$$\kappa^2 = \mu\epsilon\omega^2 + \imath\mu\sigma\omega,$$

which is satisfied by the complex wave number, $\kappa = \kappa_1 + \imath\kappa_2$,

$$\kappa_1 = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa_2 = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}.$$

Exponentially attenuated wave: $\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{-\kappa_2 x} e^{\imath(\kappa_1 x - \omega t)}$.

Note that the conductor is a dispersive medium: $\kappa_1 = \kappa_1(\omega)$.

Criterion for good conductor: $\sigma(\omega) \gg \epsilon\omega \quad \Rightarrow \ \kappa_1 \simeq \kappa_2 \simeq \sqrt{\frac{\mu\sigma\omega}{2}}$.

Characteristic penetration depth (skin depth): $\delta \doteq \frac{1}{\kappa_2} = \sqrt{\frac{2}{\mu\sigma\omega}}$.

The complex nature of the wave number κ causes a phase shift of the magnetic field $\mathbf{B}$ relative to the electric field $\mathbf{E}$ in the wave, which is most directly demonstrated by invoking Faraday's law:

$$\imath\omega\mathbf{B} = \imath\kappa\,\frac{\boldsymbol{\kappa}_1}{\kappa_1}\times\mathbf{E} \quad \Rightarrow \ \omega\mathbf{B} = \left(1+\imath\frac{\kappa_2}{\kappa_1}\right)\boldsymbol{\kappa}_1\times\mathbf{E}.$$

In a good conductor, where $\kappa_1 \simeq \kappa_2$, the phase shift is $\Delta\phi = \pi/4$.

Reflection from conductor:

Shiny metal surfaces are highly reflective. The reflected wave is largely generated by the motion of free electrons near the metal surface.

Adaptation of the result for normal-incidence reflection at the interface between dielectrics, using $\mu_1 = \mu_2 = \mu_0$, $n_1 = 1$, $n_2 = c/v_2 = c\kappa/\omega$ [lex101]:

$$\frac{E_0''}{E_0} = \frac{\mu_2 n_1 - \mu_1 n_2}{\mu_2 n_1 + \mu_1 n_2} \ \rightsquigarrow \ \frac{1-n_2}{1+n_2} = \frac{\omega - c\kappa}{\omega + c\kappa}.$$

Reflectivity: $R = \left|\frac{E_0''}{E_0}\right|^2 = \frac{(\omega - c\kappa_1)^2 + (c\kappa_2)^2}{(\omega + c\kappa_1)^2 + (c\kappa_2)^2}$.

Criteria for good conductor [lex101]: $\kappa_1 \simeq \kappa_2 \simeq \sqrt{\frac{\mu\sigma\omega}{2}}, \quad c\kappa_1 \gg \omega$.

$$\Rightarrow \ R \simeq 1 - \sqrt{\frac{8\omega\epsilon_0}{\sigma}}.$$

Dispersion:

The material parameters ϵ, μ, σ, which govern optics, vary with (angular) frequency ω, a phenomenon named *dispersion*.

Dispersion arises when the response of the optical medium to the electric or magnetic field is not instantaneous.

A classical model which illustrates this effect is the driven harmonic oscillator,

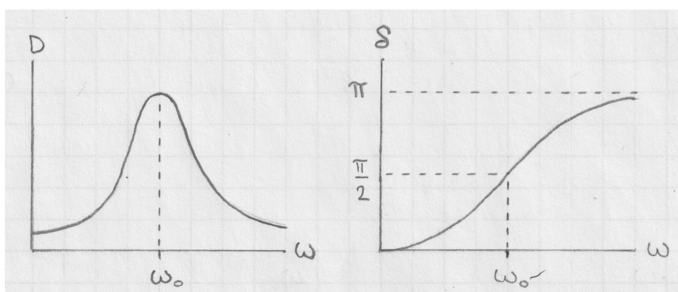
$$m\ddot{\mathbf{x}} = -K\mathbf{x} - \gamma\dot{\mathbf{x}} + \mathbf{F}(t),$$

representing an electron (mass m, charge $-e$) subject to a restoring force, $-K\mathbf{x}$, a dissipative force, $-\gamma\dot{\mathbf{x}}$, and a driving force, $\mathbf{F}(t) = -e\mathbf{E}(t)$.

Monochromatic (single-frequency) driving field: $\mathbf{E}(t) = \mathbf{E}_0 e^{-\imath\omega t}$.

Displacement of electron: $\mathbf{x}(t) = \mathbf{x}_0 e^{-\imath\omega t}, \quad \mathbf{x}_0 = \dfrac{-e\mathbf{E}_0}{K - m\omega^2 - \imath\gamma\omega}.$

The response thus modeled is frequency-dependent in both amplitude and phase, which is characteristic of dispersion [lex102]. The phase-shift is indicative of energy dissipation.



Dispersion in a dielectric:

Atomic electric dipole moment: $\mathbf{p}(t) = -e\mathbf{x}(t) = \alpha\mathbf{E}(t)$.

Atomic polarizability: $\alpha = \dfrac{e^2}{K - m\omega^2 - \imath\omega\gamma}$ (using previous expressions).

Density of atoms: ν.

Electric polarization: $\mathbf{P} = \nu\mathbf{p} = \nu\alpha\mathbf{E}$.

Displacement field: $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E}$.

Permittivity of dielectric material: $\epsilon(\omega) = \epsilon_0 + \nu\alpha(\omega)$.

Dispersion relation: $\kappa(\omega) = \omega\sqrt{\epsilon(\omega)\mu_0} \doteq \kappa_1(\omega) + \imath\kappa_2(\omega)$.

Electromagnetic wave penetrating dielectric material:

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 e^{-\kappa_2 x} e^{\imath(\kappa_1 x - \omega t)},$$

– absorption length: $d(\omega) = 1/2\kappa_2$,

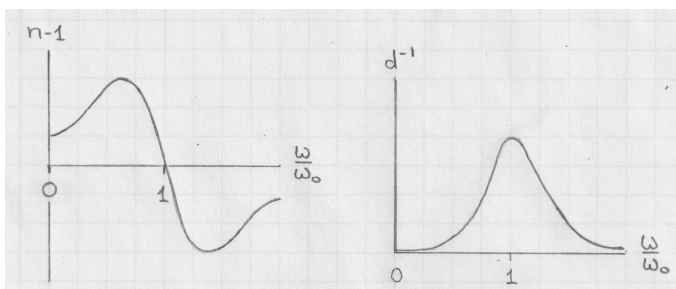
– index of refraction: $n(\omega) = c\kappa_1/\omega$,

Results for weak polarizability, $\nu\alpha \ll \epsilon_0$ [lex103]:

$$n = \frac{c\kappa_1}{\omega} = 1 + \frac{\nu e^2}{2\epsilon_0 m\omega_0^2}\frac{\omega_0^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(\omega\gamma/m)^2}, \quad \omega_0 \doteq \sqrt{\frac{K}{m}}$$

$$d^{-1} = 2\kappa_2 = \frac{\nu e^2}{\epsilon_0 mc}\frac{\omega^2\gamma/m}{(\omega_0^2-\omega^2)^2+(\omega\gamma/m)^2}.$$

Absorption peaks at resonance, $\omega \simeq \omega_0$. For $\omega < \omega_0$, $n(\omega)$ increases with ω, as evidenced in prisms and rainbows.



Dispersion in a plasma:

Plasma: gas of unbound electrons and positive ions, conducting material.

Use classical model from earlier, but with zero restoring force: $K = 0$.

Displacement of electron: $\mathbf{x}(t) = \dfrac{e}{m\omega^2 + \imath\gamma\omega}\mathbf{E}(t), \quad \mathbf{E}(t) = \mathbf{E}_0 e^{-\imath\omega t}$.

Velocity of electron: $\mathbf{v}(t) = \dfrac{d\mathbf{x}}{dt} = \dfrac{-\imath e\omega}{m\omega^2 + \imath\gamma\omega}\mathbf{E}(t)$.

Density of electrons: ν_e.

Current density: $\mathbf{J} = -e\nu_\mathrm{e}\mathbf{v} = \sigma(\omega)\mathbf{E}$.

Conductivity: $\Rightarrow \; \sigma(\omega) = \dfrac{\imath\omega e^2\nu_\mathrm{e}}{m\omega^2 + \imath\gamma\omega} \overset{\gamma\to 0}{\rightsquigarrow} \dfrac{\imath e^2\nu_\mathrm{e}}{m\omega}$.

In the dilute-plasma limit, $\gamma \to 0$. The conductivity becomes purely imaginary. Collisions are rare. Attenuation is negligibly small.

Power dissipation per volume: $\dfrac{P}{V} = \mathbf{J} \cdot \mathbf{E}$.

The imaginary $\sigma(\omega)$ in the dilute-plasma limit introduces a phase-shift of $\imath = e^{\imath\pi/2}$ between $\mathbf{J}$ and $\mathbf{E}$, implying a zero time average of $\mathbf{J} \cdot \mathbf{E}$.

Consequence: no power transfer between wave and plasma on average.

Dispersion relation of dilute-plasma wave from $\kappa^2 = \mu\epsilon\omega^2 + \imath\mu\sigma\omega$:

$$\kappa^2 = \mu_0\epsilon_0\omega^2 - \frac{\mu_0 e^2 \nu_\mathrm{e}}{m} = \frac{\omega^2 - \omega_\mathrm{p}^2}{c^2},$$

Plasma frequency: $\omega_\mathrm{p} = \sqrt{\dfrac{e^2\nu_\mathrm{e}}{m\epsilon_0}}$.

- Case $\omega > \omega_\mathrm{p}$:
  κ is real; wave propagates without attenuation, dispersion is present.
- Case $\omega < \omega_\mathrm{p}$:
  κ is imaginary; wave does not propagate through plasma; it decays exponentially on the length scale $d = c/\sqrt{\omega_\mathrm{p}^2 - \omega^2}$, no dissipation takes place; wave is reflected from plasma.

Realization of case $\omega < \omega_\mathrm{p}$: AM radio waves reflected from ionosphere.

Exercises:

▷ Anti-reflection coating [lex97]

▷ Fresnel equation for TE wave [lex98]

▷ Fresnel equation for TM wave [lex99]

▷ Electromagnetic wave in a conductor [lex100]

▷ Reflection of electromagnetic wave from a conductor [lex101]

▷ Driven harmonic oscillator: steady-state solution [lex102]

▷ Dispersion and absorption in a dielectric [lex103]