Relativity I [lln16]

Empirical facts known around 1900:

– Light is a transverse electromagnetic wave.
– The search for a medium (aether) failed.

Consequence: The speed of light, $c = 1/\sqrt{\epsilon_0\mu_0}$, as predicted by Maxwell's equations, cannot be associated with a privileged frame of reference.

Postulates of special relativity:

1. The laws of physics are the same in all inertial frames (covariance).
2. The speed of light is the same in all inertial frames (invariance).

Galileo and Newton introduced the first postulate. Einstein added the second postulate.

The conceptions of absolute space and absolute time, on which Newtonian mechanics is based, are compatible with the first postulate.

Maxwell's equations are incompatible with the first postulate, if the conceptions of absolute space and absolute time are retained.

Adding the second postulate to the first postulate requires a modified conception of (three-dimensional) space and (one-dimensional) time.

Absolute distances exist in (four-dimensional) spacetime. Maxwell's equations and relativistic mechanics are compatible with both postulates. Newtonian mechanics remains accurate for motion with $v \ll c$.

Spacetime coordinate transformations:

Consider two inertial frames in relative motion. Frame $\mathcal{F}'$ moves with constant velocity $\mathbf{v} = v\,\hat{\mathbf{i}}$ relative to frame $\mathcal{F}$. The origins coincide at $t = 0$.

Galilean transformation (satisfying postulate 1 for Newtonian mechanics):

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t. \tag{1}$$

Lorentz transformation (satisfying postulates 1 and 2 for relativistic mechanics and electromagnetism):[1]

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - vx/c^2). \tag{2}$$

Dimensionless quantities used here and later: $\beta \doteq v/c$, $\gamma \doteq 1/\sqrt{1-\beta^2}$. [lex167]

[1]Hendrik Antoon Lorentz (Dutch physicist).

Flash of light expanding:

Events are described by location and time (three space coordinates and one time coordinate). The coordinates depend on the frame of reference, (x, y, z, t) in $\mathcal{F}$ and (x', y', z', t') in $\mathcal{F}'$. The two frames coincide at $t = t' = 0$.

A light source, at rest in $\mathcal{F}$, emits a light pulse at $t = 0$ from $x = y = z = 0$.

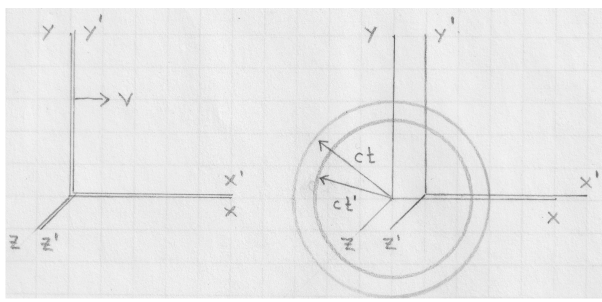
The light pulse expands radially: $x^2 + y^2 + z^2 = c^2t^2$.

What is the view from $\mathcal{F}'$?

– Galilean transformation (1) predicts:

$$(x' + vt')^2 + y'^2 + z'^2 = c^2t'^2.$$

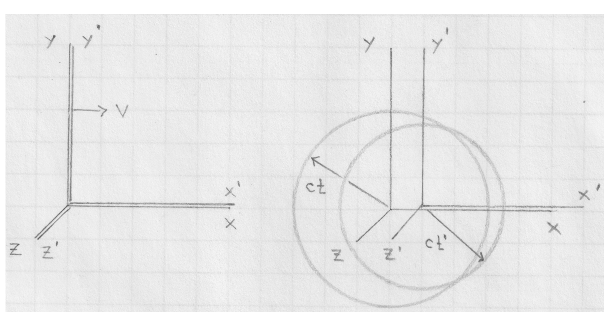
Expansion is radial from a point in relative motion to $\mathcal{F}'$.



– Lorentz transformation transformation (2) predicts:

$$\begin{aligned} x'^2 + y'^2 + z'^2 &= \gamma^2(x - vt)^2 + \underbrace{y^2 + z^2}_{c^2t^2 - x^2} \\ &= (\gamma^2 - 1)x^2 - 2\gamma^2 xvt + (\gamma^2 v^2 + c^2)t^2 \\ &= c^2\gamma^2(t - vx/c^2)^2 = c^2t'^2. \end{aligned}$$

Expansion is radial from a point at rest in $\mathcal{F}'$.



Paradigm shift:

Concepts abandoned:

- Instantaneous signals are available.
- Rigid bodies (ruler, compass) exist to map out absolute (3D) Euclidean space.
- Universal clocks exist to measure absolute (1D) time.
- Clocks can be synchronized between any two points within the same inertial system and between two points anywhere in different inertial systems.

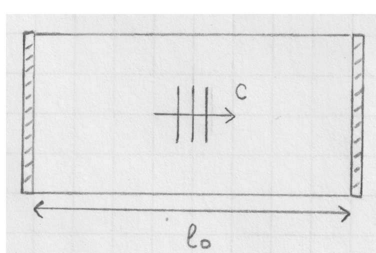
Concepts newly adopted:

- The speed of light c is universal.
- The speed of any signal cannot exceed c.
- Absolute entities exist in (4D) spacetime. Projections onto (3D) space and (1D) time are relative.
- Instruments exist to measure proper times, lengths, and angles within any inertial system,
- Clocks can be synchronized between any two points within the same inertial system, but only locally between points in relative motion.

Einstein's signature clock:

A light pulse traveling at the universal velocity c bounces between two flat mirrors a proper distance ℓ_0 apart.

Time period of the cycle: $\Delta\tau = \dfrac{2\ell_0}{c}$ (proper time interval).



Accurate clocks of any design show the same time as Einstein's signature clock, but the implications of relativistic effects may be less transparent.

Time dilation:

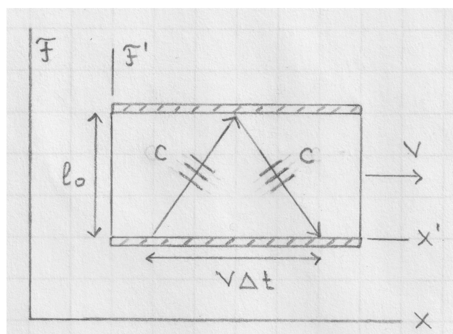
Frame $\mathcal{F}'$ moves with velocity v relative to frame $\mathcal{F}$. A clock at rest in $\mathcal{F}'$ shows proper time, $t' = \tau$.

Time elapsed in frame $\mathcal{F}$ calculated via (2):[2]

$$\Delta t = \gamma(\Delta t' + v\Delta x'/c^2) \quad \text{with} \quad \Delta x' = 0 \quad \Rightarrow \Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}}.$$

As viewed from $\mathcal{F}$ the moving clock in $\mathcal{F}'$ ticks more slowly.

Geometric interpretation of time dilation with Einstein's signature clock.



- Distance traveled by light pulse in $\mathcal{F}'$: $2\ell_0$ (proper length ℓ_0).
- Time period measured in $\mathcal{F}'$: $\Delta\tau = \dfrac{2\ell_0}{c}$ (proper time interval).
- As viewed from $\mathcal{F}$, the light pulse travels a zigzag path as shown.
- The path is longer in $\mathcal{F}$ than in $\mathcal{F}'$.
- The light pulse traverses each path at the same speed c.
- Time interval measured in $\mathcal{F}$: Δt (to be determined).
- Distance traveled by light pulse in $\mathcal{F}$: $2\sqrt{\ell_0^2 + (v\Delta t/2)^2} = c\Delta t$.
- Dilated time period measured in $\mathcal{F}$:

$$\Rightarrow \Delta t = \frac{2\ell_0/c}{\sqrt{1-v^2/c^2}} = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}} > \Delta\tau.$$

- A clock in $\mathcal{F}'$ viewed from $\mathcal{F}$ ticks more slowly than a clock in $\mathcal{F}$. [lex90]

[2]Primed and unprimed variables can be interchanged in (2) if v is replaced by $-v$.

Length contraction:

Frame $\mathcal{F}'$ again moves with velocity v relative to frame $\mathcal{F}$. A rod at rest in $\mathcal{F}'$ has proper length, $\Delta x' = \ell_0$.

The measurement of a length implies the simultaneous measurement of two positions in the same frame.

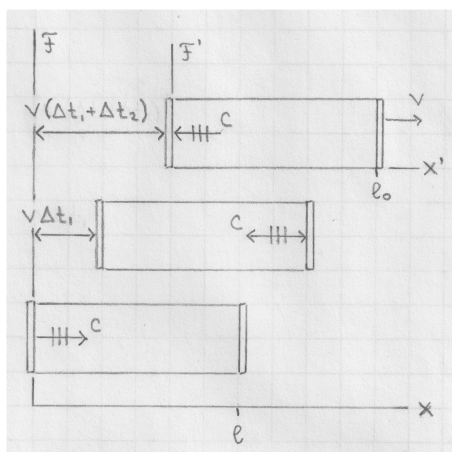
Positions transformed from $\mathcal{F}'$ to $\mathcal{F}$ via (2): $\Delta x = \gamma \overbrace{\Delta x'}^{\ell_0} + \gamma v \Delta t'$.

Simultaneity in $\mathcal{F}$ imposed using (2): $\Delta t = \gamma(\Delta t' + v\Delta x'/c^2) = 0$.

$$\Rightarrow \quad \Delta t' = -\frac{v\Delta x'}{c^2} = -\frac{v\ell_0}{c^2} \quad \Rightarrow \quad \gamma v \Delta t' = -\gamma \ell_0 \frac{v^2}{c^2}.$$

Contracted length in $\mathcal{F}$: $\Delta x = \gamma \ell_0 \left(1 - \dfrac{v^2}{c^2}\right) = \sqrt{1 - v^2/c^2}\, \ell_0$.

Geometric interpretation of length contraction with Einstein's signature clock.



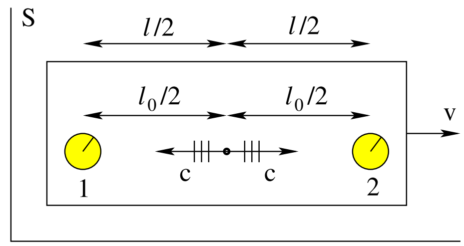
[lex92][lex145][lex160][lex161][lex162]

- Forward path in $\mathcal{F}$ is prolonged: $c\Delta t_1 = \ell + v\Delta t_1 \quad \Rightarrow \quad \Delta t_1 = \dfrac{\ell}{c - v}$.

  Reverse path in $\mathcal{F}$ is shortened: $c\Delta t_2 = \ell - v\Delta t_2 \quad \Rightarrow \quad \Delta t_2 = \dfrac{\ell}{c + v}$.
- Time period measured in $\mathcal{F}$:
$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{\ell}{c - v} + \frac{\ell}{c + v} = \frac{2\ell/c}{1 - v^2/c^2}.$$
- Relation to proper time period: $\Delta t = \dfrac{\Delta\tau}{\sqrt{1 - v^2/c^2}} = \dfrac{2\ell_0/c}{\sqrt{1 - v^2/c^2}}$.
- Length compared with proper length: $\ell = \ell_0\sqrt{1 - v^2/c^2}$.

Relativity of simultaneity:

Frame S', which moves with velocity v relative to frame S, has two synchronized clocks 1 and 2 separated from each other a proper distance ℓ_0.



A light signal is emitted at $t = t' = 0$ midway between the clocks.

Signal arrival times in S': $ct'_1 = ct'_2 = \ell_0/2 \quad \Rightarrow \Delta\tau = t'_2 - t'_1 = 0$.

Signal arrival times in S: $ct_1 = \ell/2 - vt_1, \quad ct_2 = \ell/2 + vt_2$.

$$\Rightarrow \Delta t = t_2 - t_1 = \frac{\ell/2}{c - v} - \frac{\ell/2}{c + v} = \frac{\ell v/c^2}{1 - v^2/c^2} = \frac{\ell_0 v/c^2}{\sqrt{1 - v^2/c^2}}.$$

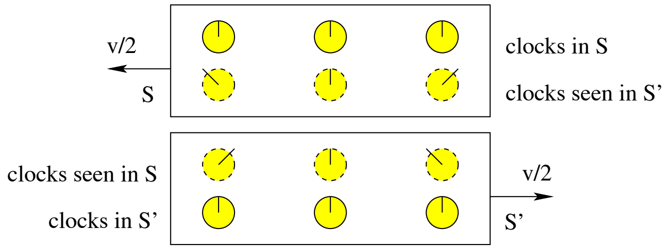
Time difference measured with clock in S translated to clock in S':

$$\Delta t' = \Delta t\sqrt{1 - v^2/c^2} = \frac{\ell_0 v}{c^2}.$$

The result is in conflict with measurement $\Delta\tau = 0$ made in S'.

Conclusion: Synchronized clocks in S' are not synchronized when viewed from S: clock 1, a proper distance ℓ_0 behind clock 2, is ahead in time by $\ell_0 v/c^2$.

Consider two arrays of synchronized clocks in relative motion as shown.



When two observers at different positions in S read the time of synchronized clocks in S' simultaneously according to clocks in S, the readings are different.

The moving clock ahead in space will show a later time than the clock behind in space. The same is true when the roles of frames S and S' are interchanged.

Time dilation paradox resolved:

Two reference frames S and S' are in relative motion. An observer in S determines that time in S' is slowed, whereas an observer in S' determines that time in S is slowed. How can both observers be right?

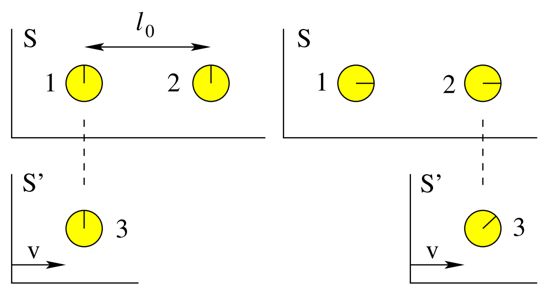
Clocks 1 and 2 in S are synchronized in S for all times. Clock 1 in S is synchronized with clock 3 in S' at $t = 0$, when they face each other.

View from frame S:

Distance between clocks 1 and 2: ℓ_0 (proper distance).

Reading of clock 2 when clock 3 arrives there: $t = \ell_0/v$.

Time elapsed in S': $t' = (\ell_0/v)\sqrt{1 - v^2/c^2}$ (clock 3 slowed).



View from frame S':

Distance between clocks 1 and 2: $\ell' = \ell_0\sqrt{1 - v^2/c^2}$ (contracted).

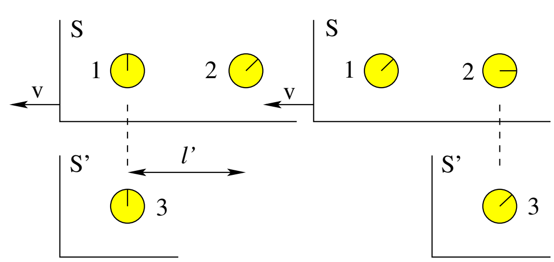
Reading of clock 3 when it faces clock 2: $t' = \ell'/v = (\ell_0/v)\sqrt{1 - v^2/c^2}$.

Dilated time elapsed in S when clock 2 faces clock 3 is slowed:

$$\Delta t = t'\sqrt{1 - v^2/c^2} = (\ell_0/v)(1 - v^2/c^2).$$

Initial reading of clock 2 as viewed from S' is different from that of clock 1. Clock 2 is ahead of clock 1 by $t_i = \ell_0 v/c^2$.

Final reading of clock 2: $t_i + \Delta t = \ell_0/v$ (consistent with view form S).



Length contraction paradox resolved:

Two frames S and S' are in relative motion with velocity v. Jack is at rest in S. Jill is at rest in S'.

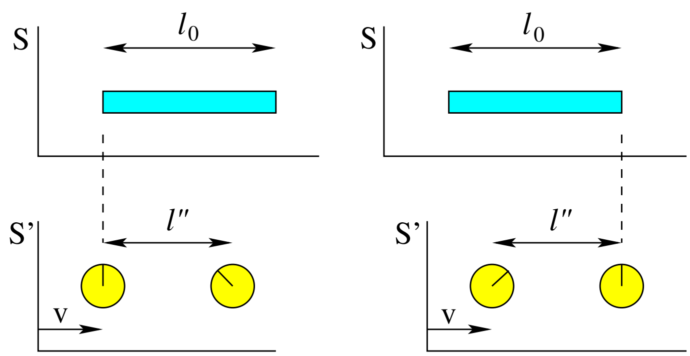
Jack is holding a rod of proper length ℓ_0. Jill measures Jack's rod by holding her hands a (contracted) distance $\ell' = \ell_0\sqrt{1 - v^2/c^2}$ apart.

Jack, in turn, measures the distance between Jill's hands, $\ell'' = \ell'\sqrt{1 - v^2/c^2}$, which is further contracted and thus falls short of ℓ_0, his direct measurement.

What did we miss? The relativity of simultaneity!

Let Jill mark her measurement by two clocks synchronized in S' coinciding with the endpoints of Jack's rod.

From Jill's perspective, the distance between the (synchronized) clocks is $\ell' = \ell_0\sqrt{1 - v^2/c^2}$. Jack's rod has a contracted length in S'.



From Jack's perspective (see diagram), the distance between the clocks is $\ell'' = \ell_0(1 - v^2/c^2)$ and the clocks are out of sync by $\Delta t' = \ell' v/c^2$ in S'.

This time lag seen by Jack on Jill's clocks is dilated relative to clocks in S to $\Delta t = \Delta t'/\sqrt{1 - v^2/c^2} = \ell_0 v/c^2$.

If Jack wants to determine the length of his own rod via Jill's measurement, he must mark the positions of her clocks at instances when they record the same time as seen from S.

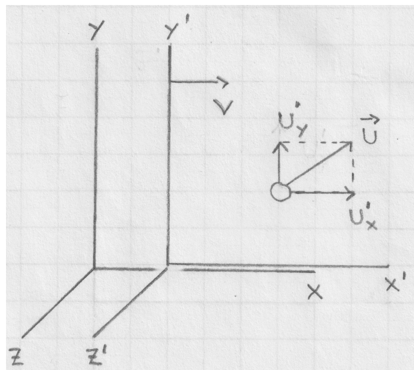
The result is in agreement with the proper length of Jack's rod:

$$\ell'' + v\Delta t = \ell_0(1 - v^2/c^2) + \ell_0 v^2/c^2 = \ell_0.$$

Addition of velocities:

A particle moves in frame $\mathcal{F}'$, which, in turn, moves relative to frame $\mathcal{F}$.

With no loss of generality we assume (i) that frame $\mathcal{F}'$ moves with velocity v in x-direction relative to frame $\mathcal{F}$ and (ii) that the motion of the particle in $\mathcal{F}'$ is in the xy-plane: $\mathbf{u}' = u'_x\hat{\mathbf{i}} + u'_y\hat{\mathbf{j}}$.



Longitudinal velocity addition:

$$u_x \doteq \frac{dx_p}{dt} = \frac{\gamma(dx'_p + v dt')}{\gamma(dt' + v dx'_p/c^2)} = \frac{dx'_p/dt' + v}{1 + v(dx'_p/dt')/c^2} = \frac{u'_x + v}{1 + u'_x v/c^2} < c.$$

Transverse velocity addition:

$$u_y \doteq \frac{dy_p}{dt} = \frac{dy'_p}{\gamma(dt' + v dx'_p/c^2)} = \frac{dy'_p/dt'}{\gamma[1 + v(dx'_p/dt')/c^2]} = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}.$$

In these derivations we have mainly used the Lorentz transformation of space and time intervals. [lex168]

- The case of a particle at rest in $\mathcal{F}'$:

$$u'_x = u'_y = 0 \quad \Rightarrow \; u_x = v, \quad u_y = 0.$$

- The case of a photon moving longitudinally in $\mathcal{F}'$:

$$u'_x = c, \quad u'_y = 0 \quad \Rightarrow \; u_x = \frac{c + v}{1 + cv/c^2} = c, \quad u_y = 0.$$

- The case of a photon moving transversally in $\mathcal{F}'$:

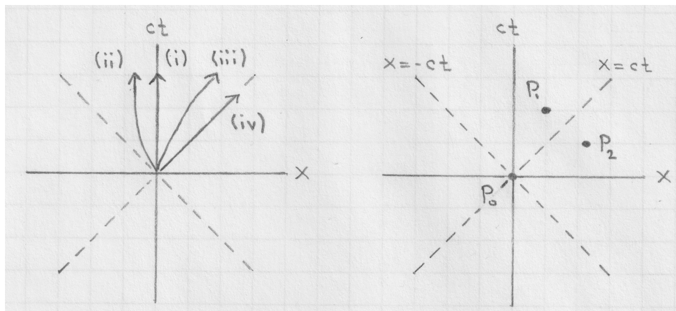
$$u'_x = 0, \quad u'_y = c \quad \Rightarrow \; u_x = v, \quad u_y = c\sqrt{1 - v^2/c^2},$$

$$\Rightarrow \; |\mathbf{u}| = \sqrt{u_x^2 + u_y^2} = c.$$

Events and worldlines:

Events are points (worldpoints) in 4-dimensional spacetime.

Worldlines are trajectories of physical objects in spacetime: (i) particle at rest; (ii) particle slowing down; (iii) particle speeding up; (iv) photon cruising.



World lines of massive objects starting from here ($x = 0$) at present ($t = 0$) are confined to the inside of the (upper) future light cone. They all have a history as world lines inside the (lower) past light cone.

A light pulse emitted here and now travels along the future light cone. All light pulses traveling along the past light cone are received here and now.

Distances and time intervals generally vary between frames:

$$\sqrt{(dx')^2 + (dy')^2 + (dz')^2} \neq \sqrt{(dx)^2 + (dy)^2 + (dz)^2}, \quad dt' \neq dt.$$

The spacetime "distance" defined by its square as

$$(ds)^2 \doteq (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2,$$

does not vary between frames. It is invariant under Lorentz transformations:

$$\begin{aligned}
(ds')^2 &= (dx')^2 + (dy')^2 + (dz')^2 - c^2(dt')^2 \\
&= (dy)^2 + (dz)^2 + \gamma^2\left[(dx - vdt)^2 - c^2(dt - (v/c^2)dx)^2\right] \\
&= (dy)^2 + (dz)^2 + \gamma^2\left[[(dx)^2(1 - v^2/c^2) - c^2(dt)^2(1 - v^2/c^2)\right] \\
&= (dy)^2 + (dz)^2 + (dx)^2 - c^2(dt)^2 = (ds)^2.
\end{aligned}$$

where we have used

$$dx' = \gamma(dx - vdt), \quad dt' = \gamma(dt - vdx/c^2), \quad 1 - v^2/c^2 = 1/\gamma^2.$$

The invariant $(ds)^2$, which can be positive, negative, or zero, determines the nature of the relation between events.

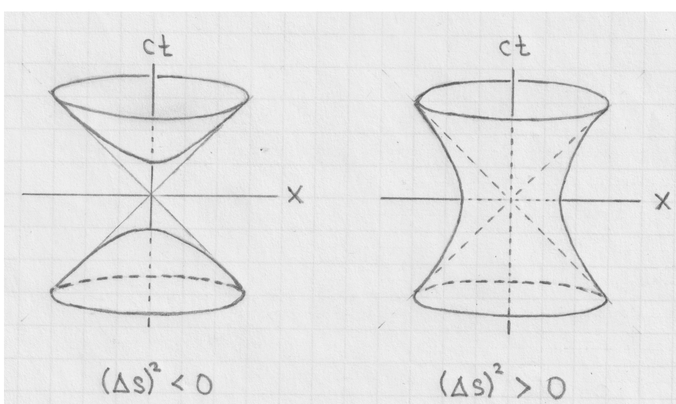
- Events P_0 and P_1 have $(\Delta s)^2 < 0$, implying a *time-like* relation. Such events can be (unambiguously) causally related and are at the same position in some reference frame.
- Events P_0 and P_2 have $(\Delta s)^2 > 0$, implying a *space-like* relation. Such events are not causally related and are simultaneous in some reference frame.
- Any event on the light cone has a *light-like* relation to the event at the apex (here and now).

The drawing on the left shows a two-sheet hyperboloid, $(\Delta s)^2 = \text{const} < 0$.

- It contains the possible locations of two events which are time-like related to the here and now (one past event and one future event) as viewed from different inertial systems.
- The future (past) event is on the upper (lower) sheet.
- There exist inertial systems for which either event takes place here.
- In no inertial system does either event take place now.

The drawing on the right shows a one-sheet hyperboloid, $(\Delta s)^2 = \text{const} > 0$.

- It contains the possible locations of an event which is space-like related to the here and now as viewed from different inertial systems.
- In no inertial system does the event take place here.
- There exists an inertial system for which the event takes place now.



Mass and energy:

The postulates of Newtonian mechanics imply separate conservation laws for the total energy and the total mass in an autonomous system. These conclusions are incompatible with relativistic mechanics.

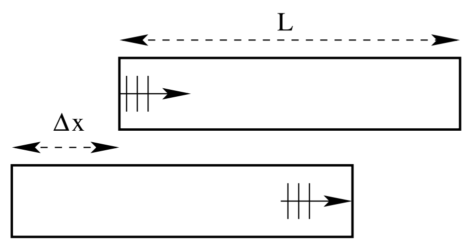
[lln15] We know that electromagnetic fields carry energy and momentum. Earlier we expressed energy and momentum density of an electromagnetic wave in terms of the Poynting vector as follows:

$$u = \frac{|\mathbf{S}|}{c}, \quad |\mathbf{p}| = \frac{|\mathbf{S}|}{c^2}, \quad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}.$$

From this and the superposition principle follows a characteristic relation between the energy E and the momentum p (magnitude) of a light pulse:

$$E = pc.$$

Consider a macroscopic rectangular box of mass M and length L at rest in the inertial frame of the observer. A light pulse of momentum p and energy $E = pc$ is emitted at one end and absorbed at the other end as shown.



According to Newtonian mechanics, energy is transferred from one end to the other end of the box, but no mass is transferred.

The box recoils a distance Δx during the time of flight Δt of the light pulse in accordance with momentum conservation.

$$Mv = -\frac{E}{c}, \quad \Delta x = v\Delta t, \quad \Delta t = \frac{L}{c} \quad \Rightarrow \; \Delta x = -\frac{EL}{Mc^2}.$$

The absence of external forces requires the center of mass, initially at rest, to remain at the same (rest) position. Therefore, the nonzero displacement $\Delta x < 0$ of the box is incompatible with zero mass transfer.

To reconcile momentum conservation with a zero shift of the center of mass of the box, we must attribute a mass m to the energy transferred form one end to the other of the box:

$$\Delta x_{\rm cm} = 0 \quad \Rightarrow \; m(L + \Delta x) + (M - m)\Delta x = 0.$$

In the above equation (with $\Delta x < 0$),

> the first parenthesis accounts for the shortened distance traveled by the light pulse,
>
> the second parenthesis accounts for the reduced mass of the box.

The no-shift condition for the center of mass thus requires

$$\Delta x = -\frac{mL}{M}.$$

When set equal to the earlier expression for Δx, we arrives at Einstein's famous relation for the energy-equivalence of mass:

$$E = mc^2.$$

Photons are massless but carry energy and momentum. Atoms which emit (absorb) photons lose (gain) mass and recoil in both processes.

[lln24] The photon energy and momentum are established in quantum mechanics:

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}, \quad \frac{\omega}{|\mathbf{k}|} = c.$$

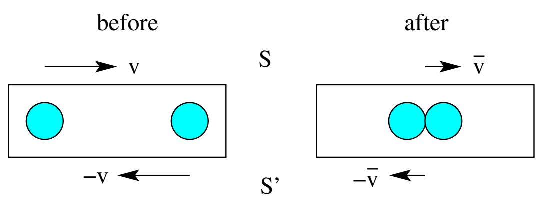
A light pulse is a bunch of photons with different (angular) frequencies ω, all traveling at the speed c.

Relativistic momentum:

Two particles of equal mass m as measured when at rest are undergoing an inelastic collision. Frame S' moves with velocity v relative to frame S.

In frame S a particle moves with velocity v toward a particle at rest. In frame S', the first particle is at rest and the second particle moves with velocity $-v$ toward it.

Note the dual role of v as the initial velocity of one particle in each frame and as the relative velocity between frames.



The goal here is to construct an expression for the momentum of massive particle, consistent with momentum conservation in this collision process.

Ansatz for relativistic momentum:

$$\mathbf{p} = \tilde{m}(v)\mathbf{v}, \quad \tilde{m}(0) = m.$$

The auxiliary quantity $\tilde{m}(v)$, named *relativistic mass*, is only used in introductory contexts such as here. The term "mass" usually means "rest mass" m. Relativistic modifications are then assigned to energy and momentum.

Conservation of total momentum with two masses m before the collision and one mass M after the collision implies the following relation (in frame S):

$$\tilde{m}(v)v + \tilde{m}(0)0 = \tilde{M}(\bar{v})\bar{v}.$$

Use the longitudinal velocity addition rule (derived earlier) to determine the relation between v and $\bar{v}$. For this purpose set $u_x = \bar{v}$ and $u'_x = -\bar{v}$:

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} \quad \Rightarrow \quad \bar{v} = \frac{-\bar{v} + v}{1 - \bar{v}v/c^2} \quad \Rightarrow \quad v = \frac{2\bar{v}}{1 + \bar{v}^2/c^2}.$$

Lorentz invariance of momentum conservation under transverse motion implies a second relation between m and M during the collision: [lex169]

$$\tilde{M}(\bar{v}) = \tilde{m}(v) + \tilde{m}(0).$$

From the two relations and the velocity addition rule, the following expression for the auxiliary quantity named *relativistic mass* can be inferred:

$$\tilde{m}(v) = \frac{\tilde{m}(0)}{\sqrt{1 - v^2/c^2}}.$$

[lex170]

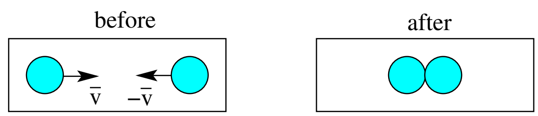
The expression for the relativistic momentum then follows directly:

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}.$$

Note that the speed of a massive particle is limited by speed of light ($v < c$), but its momentum has no upper limit.

Relativistic energy:

Here we observe the inelastic collision between two particles with equal rest mass m from the center-of-mass frame. The total momentum vanishes.



Relativistic mass is conserved in this collision, a conclusion inferred earlier from momentum conservation:

$$\tilde{M}(0) = \tilde{m}(\bar{v}) + \tilde{m}(-\bar{v}) = 2\tilde{m}(\bar{v}) = \frac{2m}{\sqrt{1-\bar{v}^2/c^2}}.$$

Increase in rest mass during this collision:

$$\Delta M = \tilde{M}(0) - 2m = 2m\left(\frac{1}{\sqrt{1-\bar{v}^2/c^2}} - 1\right).$$

General expression for the relativistic energy:

$$E \doteq \tilde{m}(v)c^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}.$$

Note that the energy of a particle increases without bound as its velocity approaches the limiting value c.

Conservation of relativistic energy is more general than conservation of relativistic mass. In this instance they are equivalent:

$$\Delta E = \tilde{M}(0)c^2 - 2\tilde{m}(\bar{v})c^2 = 0.$$

Kinetic energy of a particle at velocity v:

$$T \doteq E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) \overset{v\ll c}{\rightsquigarrow} \frac{1}{2}mv^2.$$

The inelastic collision viewed from the center-of-mass frame converts all kinetic energy into additional rest mass: [lex93][lex94]
[lex165]

$$\Delta Mc^2 = 2T = 2mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) \overset{v\ll c}{\rightsquigarrow} 2\left(\frac{1}{2}m\bar{v}^2\right).$$

When two macroscopic objects collide, part of the kinetic energy is converted into heat. Heat includes forms of microscopic kinetic energy. The (relativistic) mass increase of particles in thermal agitation contributes to ΔM.

We know from Newtonian mechanics that the relation between momentum and kinetic energy of a massive particle is

$$T = \frac{p^2}{2m},$$

which we infer from the familiar expressions,

$$p = mv, \quad T = \frac{1}{2}mv^2.$$

From the relativistic expressions (derived previously) for energy E and momentum p (magnitude) of a massive particle,

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad p = \frac{mv}{\sqrt{1 - v^2/c^2}},$$

we can infer the energy-momentum relation as follows:

$$E^2 - (pc)^2 = \frac{m^2c^4 - m^2c^2v^2}{1 - v^2/c^2} = m^2c^4 \quad \Rightarrow \; E = \sqrt{p^2c^2 + m^2c^4}.$$

The nonrelativistic limit splits the relativistic energy into rest energy and kinetic energy:

$$E = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \;\xrightarrow{p \ll mc}\; mc^2 + \frac{p^2}{2m}.$$