Electrodynamics II [lln15]

Maxwell's equations, quite generally, express four local linear relations between fields and sources. They include Gauss's laws for the electric field and the magnetic field, and two laws that couple fields dynamically: Faraday's law and Ampère's law.

In the presence of magnetizable and/or electrically polarizable matter, an adequate description increases the number of fields and sources with additional relations between them.

Maxwell's equations in vacuum:

For now we consider two fields and two ideal sources:

- electric field $\mathbf{E}(\mathbf{x}, t)$ [N/C],
- magnetic field $\mathbf{B}(\mathbf{x}, t)$ [T]=[N/Am],
- charge density $\rho(\mathbf{x}, t)$ [C/m^3],
- current density $\mathbf{J}(\mathbf{x}, t)$ [A/m^2].

Charge conservation requires that the sources satisfy the continuity equation,

$$\frac{d}{dt}\int_V d^3x\,\rho = -\oint_S d\mathbf{a}\cdot\mathbf{J} = -\int_V d^3x\,\nabla\cdot\mathbf{J} \quad\Rightarrow\quad \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}. \tag{1}$$

The four Maxwell's equations in differential form (with $\epsilon_0\mu_0c^2 = 1$) read,

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{3}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{4}$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0(\mathbf{J} + \mathbf{J}_D). \tag{5}$$

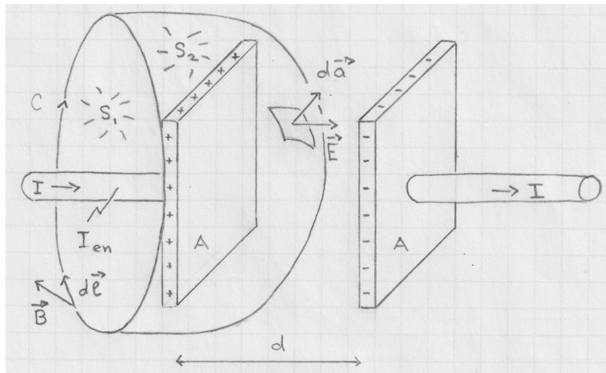
Displacement current density: $\mathbf{J}_D = \epsilon_0\,\dfrac{\partial\mathbf{E}}{\partial t}$.

Integral form of Maxwell's equations:

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{in}}}{\epsilon_0}, \qquad \oint_S \mathbf{B}\cdot d\mathbf{a} = 0,$$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{a}, \quad \oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0\int_S (\mathbf{J} + \mathbf{J}_D)\cdot d\mathbf{a}.$$

Displacement current illustrated:



In the setup shown a capacitor is being charged up by a conduction current I flowing in both connecting wires. No conduction current flows between the plates, but an electric field is being built up.

If we consider the loop C as the perimeter of the surface S_1, the integral version of Ampère's law reads:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{S_1} \mathbf{J} \cdot d\mathbf{a} = \mu_0 I.$$

The same loop C is also the perimeter of surface S_2. Ampère's law now reads:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{S_2} \mathbf{J}_D \cdot d\mathbf{a} = \mu_0 I_D.$$

It is straightforward to show that $I = I_D$ for a parallel-plate capacitor. [lex82]

Two self-consistency checks:

– The mathematical identity, $\nabla \cdot (\nabla \times \mathbf{E}) = 0$, is satisfied if we combine (3), (4) and commute time and space derivatives.

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} \underbrace{(\nabla \cdot \mathbf{B})}_{0} = 0.$$

– The mathematical identity, $\nabla \cdot (\nabla \times \mathbf{B}) = 0$, is satisfied if we combine (2), (5), (1) and commute time and space derivatives.

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \underbrace{(\nabla \cdot \mathbf{E})}_{\rho/\epsilon_0} = \mu_0 \underbrace{\left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right)}_{0} = 0.$$

Scalar potential and vector potential:

Any irrotational (curl-free) field can be expressed as the gradient of a scalar potential. The electric field $\mathbf{E}$ was irrotational in electrostatics.

[lln5] [lln12] Any solenoidal (divergence-free) field can be expressed as the curl of a vector potential. The magnetic field $\mathbf{B}$ was solenoidal in magnetostatics.

In electrodynamics, the magnetic field $\mathbf{B}(\mathbf{x}, t)$ remains solenoidal owing to (3). It can be expressed as the curl of the vector potential $\mathbf{A}(\mathbf{x}, t)$:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{6}$$

The electric field $\mathbf{E}(\mathbf{x}, t)$ is no longer irrotational, but an irrotational field, expressible as the gradient of a scalar potential $\Phi(\mathbf{x}, t)$, can be constructed from (4) in combination with (6):

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{A} \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \quad \Rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi.$$

The electric field in an electrodynamics context can thus be derived from the scalar and vector potentials as follows:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{7}$$

Maxwell's equations can be consolidated into equations for the scalar potential $\Phi(\mathbf{x}, t)$ and the vector potential $\mathbf{A}(\mathbf{x}, t)$: [lex83]

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}, \tag{8}$$

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial\Phi}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{9}$$

Note the difference between the derivatives $\nabla(\nabla \cdot \mathbf{A})$ and $\nabla^2\mathbf{A} = (\nabla \cdot \nabla)\mathbf{A}$.

- The divergence applied to the vector $\mathbf{A}$ produces a scalar and the gradient applied to that scalar produces a vector again.
- The (Laplacian) operator ∇^2 acts on each (Cartesian) component of $\mathbf{A}$ separately, producing the components of a vector in a different way.
- For vectors $\mathbf{A}$ expressed in curvilinear coordinates, we can infer $\nabla^2\mathbf{A}$ by use of the identity: [gmd1-A]

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}.$$

Gauge invariance:

The solution of Maxwell's equations (2)-(5) for sources that satisfy the continuity equation (1) is unique if boundary conditions are properly specified. However, the scalar and vector potentials are not uniquely determined.

The magnetic field $\mathbf{B}(\mathbf{x},t)$ and the electric field $\mathbf{E}(\mathbf{x},t)$ derived via (6) and (7), respectively, remain invariant if the scalar potential $\Phi(\mathbf{x},t)$ and the vector potential $\mathbf{A}(\mathbf{x},t)$ undergo a gauge transformation as follows:

$$\mathbf{A} \;\to\; \mathbf{A}' = \mathbf{A} + \nabla f, \quad \Phi \;\to\; \Phi' = \Phi - \frac{\partial f}{\partial t},$$

where $f(\mathbf{x},t)$ is an arbitrary (differentiable) scalar function. [lex83]

The theory of electromagnetism as established by Eqs. (1)-(5) is thus said to be *gauge invariant*. The two potentials can be fixed by the choice of a gauge condition. Two common choices are the following.

Coulomb gauge condition: $\nabla \cdot \mathbf{A} = 0$.

Consequence: $\nabla^2\Phi = -\dfrac{\rho}{\epsilon_0} \quad \Rightarrow \; \Phi(\mathbf{x},t) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int d^3x'\,\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$.

The Poisson equation and its formal solution, familiar from electrostatics, remain valid. The expression for $\Phi(\mathbf{x},t)$ mimics an instantaneous relation between potential and source at different points in space. The expression is formally correct, but the interpretation is more subtle. [lex134][lex135]

Lorenz gauge condition:[1] $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0\,\dfrac{\partial \Phi}{\partial t}$.

Consequence: The equations that relate potentials and sources decouple and have similar forms:

$$-\nabla^2\Phi + \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = \frac{\rho}{\epsilon_0}, \quad -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}.$$

The Lorenz gauge is convenient for the description of radiation processes. Both potentials satisfy (inhomogeneous) wave equations.

[1] Ludvig Lorenz, Danish physicist.

Maxwell's equations in matter:

List of relevant quantities:

- electric field: $\mathbf{E}(\mathbf{x},t)\quad$ [N/C],
- electric polarization: $\mathbf{P}(\mathbf{x},t)\quad$ [C/m^2],
- displacement field: $\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)+\mathbf{P}(\mathbf{x},t)\quad$ [C/m^2],
- charge density: $\rho(\mathbf{x},t)=\rho_{\rm f}(\mathbf{x},t)+\rho_{\rm b}(\mathbf{x},t)\quad$ [C/m^3],
- free charge density: $\rho_{\rm f}(\mathbf{x},t)\quad$ [C/m^3],
- bound charge density: $\rho_{\rm b}(\mathbf{x},t)=-\nabla\cdot\mathbf{P}(\mathbf{x},t)\quad$ [C/m^3],
- magnetic induction: $\mathbf{B}(\mathbf{x},t)\quad$ [T]=[N/Am],
- magnetization: $\mathbf{M}(\mathbf{x},t)\quad$ [A/m],
- magnetic field: $\mathbf{H}(\mathbf{x},t)=\dfrac{1}{\mu_0}\mathbf{B}(\mathbf{x},t)-\mathbf{M}(\mathbf{x},t)\quad$ [A/m],
- current density: $\mathbf{J}(\mathbf{x},t)=\mathbf{J}_{\rm f}(\mathbf{x},t)+\mathbf{J}_{\rm b}(\mathbf{x},t)+\mathbf{J}_{\rm P}(\mathbf{x},t)\quad$ [A/m^2],
- free current density: $\mathbf{J}_{\rm f}(\mathbf{x},t)\quad$ [A/m^2],
- bound current density: $\mathbf{J}_{\rm b}(\mathbf{x},t)=\nabla\times\mathbf{M}(\mathbf{x},t)\quad$ [A/m^2],
- polarization current density: $\mathbf{J}_{\rm P}(\mathbf{x},t)=\dfrac{\partial}{\partial t}\mathbf{P}(\mathbf{x},t)\quad$ [A/m^2],

Note that $\rho_{\rm b}$ and $\mathbf{J}_{\rm P}$ are associated with electric polarization $\mathbf{P}$, whereas $\mathbf{J}_{\rm b}$ is associated with magnetization $\mathbf{M}$.

Maxwell's equations in vacuum, (2)-(5), remain valid in the presence of polarizable and magnetizable matter. Their usefulness is diminished for lack of a priori knowledge of some source terms, specifically $\rho_{\rm b}$, $\mathbf{J}_{\rm b}$, $\mathbf{J}_{\rm P}$.

The two Maxwell equations (2) and (5), which couple fields to sources, are restated for the fields $\mathbf{D}$ and $\mathbf{H}$ such as to explicitly contain only free charge density $\rho_{\rm f}$ and free current density $\mathbf{J}_{\rm f}$.

$$\nabla\cdot\mathbf{D}=\rho_{\rm f}, \tag{10}$$

$$\nabla\cdot\mathbf{B}=0, \tag{11}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \tag{12}$$

$$\nabla\times\mathbf{H}=\mathbf{J}_{\rm f}+\frac{\partial\mathbf{D}}{\partial t}. \tag{13}$$

Restatement [(2) $\to$ (10)] of Gauss's law for the electric field:

$$\epsilon_0\nabla\cdot\mathbf{E}=\rho_{\rm f}+\rho_{\rm b},\quad \mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P},\quad \rho_{\rm b}=-\nabla\cdot\mathbf{P}\quad\Rightarrow\ \nabla\cdot\mathbf{D}=\rho_{\rm f}.$$

Restatement [(5) → (13)] of Ampère's law:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_\mathrm{f} + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

$$\Rightarrow \ \nabla \times \underbrace{\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right)}_{\mathbf{H}} = \mathbf{J}_\mathrm{f} + \frac{\partial}{\partial t} \underbrace{(\epsilon_0 \mathbf{E} + \mathbf{P})}_{\mathbf{D}}.$$

The continuity equation (1) remains valid. The free charge and current densities satisfy a separate continuity equation. The bound current density is divergence free. It follows that the bound charge density and the polarization current density satisfy a separate continuity equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \quad \nabla \cdot \mathbf{J}_\mathrm{f} = -\frac{\partial \rho_\mathrm{f}}{\partial t}, \quad \nabla \cdot \mathbf{J}_\mathrm{b} = 0 \quad \Rightarrow \ \nabla \cdot \mathbf{J}_\mathrm{P} = -\frac{\partial \rho_\mathrm{b}}{\partial t}.$$

The benefit of the restated Maxwell equations is that data about the remaining source terms ρ_f and $\mathbf{J}_\mathrm{f}$ are more readily available.

The price to be paid for this benefit is that the restated Maxwell equations now include the matter vector fields $\mathbf{P}(\mathbf{x}, t)$ and $\mathbf{M}(\mathbf{x}, t)$, contained in the definitions of $\mathbf{D}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$.

The dependence of the matter fields, $\mathbf{P}(\mathbf{x}, t)$ and $\mathbf{M}(\mathbf{x}, t)$, on the fundamental fields, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, is expressed in constitutive equations.

Regime of linear and decoupled responses:

- ▷ $\mathbf{P} = \epsilon_0 \chi_\mathrm{e} \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E}, \quad \epsilon = \epsilon_0(1 + \chi_\mathrm{e}) = \kappa \epsilon_0,$
- ▷ $\mathbf{M} = \chi_\mathrm{m} \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mu = \mu_0(1 + \chi_\mathrm{m}) = \kappa_\mathrm{m} \mu_0,$

The boundary conditions at interfaces between media with permittivities ϵ_1, ϵ_2 and permeabilities μ_1, μ_2 are still valid.

Interface conditions for the normal parts of two fields, where σ_f is the free interface charge density:

$$\mathbf{D}_{2\perp} - \mathbf{D}_{1\perp} = \sigma_\mathrm{f}, \quad \mathbf{B}_{2\perp} - \mathbf{B}_{1\perp} = 0.$$

Interface conditions for the tangential parts of two fields, where $\mathbf{K}_\mathrm{f}$ is the free interface current density:

$$\mathbf{E}_{2\|} - \mathbf{E}_{1\|} = 0, \quad \mathbf{H}_{2\|} - \mathbf{H}_{1\|} = \mathbf{K}_\mathrm{f} \times \hat{\mathbf{n}}.$$

The normal unit vector $\hat{\mathbf{n}}$ in the convention adopted here points from medium 1 to medium 2. [lex140]

Path from restricted scenarios toward generality:

In these course materials, we have begun with restrictive conditions and then gradually relaxed them selectively.

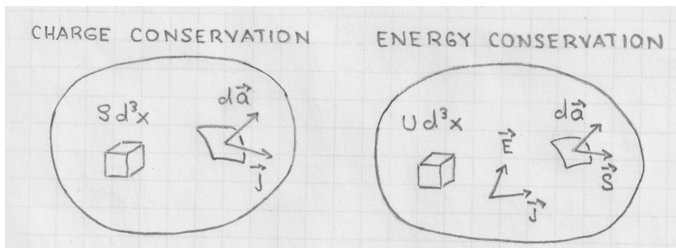
- *Electrostatics with ideal sources.*
  Static electric field $\mathbf{E}$ generated by free charge density $\rho_{\rm f}$ of immobile charge carriers or mobile charge carriers at equilibrium. Scalar potential Φ. Electrostatic force on charged particles.
- *Electrostatics with dielectrics.*
  Bound charge density $\rho_{\rm b}$ induced by static electric field. Electric polarization $\mathbf{P}$ from induced atomic electric dipoles or permanent molecular electric dipoles. Linear dielectric response. Displacement field $\mathbf{D}$.
- *Steady electric currents.*
  Divergence-free current density $\mathbf{J}_{\rm f}$ of mobile charge carriers caused in conducting materials by static electric fields. Ohm's law as another form of linear response.
- *Magnetostatics with ideal sources.*
  Static magnetic field $\mathbf{B}$ generated by steady and divergence-free currents. Vector potential $\mathbf{A}$. Magnetic force acting on moving charged particles or current-carrying conductors.
- *Magnetostatics of magnetizable matter.*
  Magnetization $\mathbf{M}$ from electron spins or electron orbital magnetic moments. Linear magnetic response. $\mathbf{H}$-field. Bound current $\mathbf{J}_{\rm b}$ from magnetization.
- *Electrodynamics with ideal sources.*
  Maxwell field equations in vacuum with ideal sources (free charge and current densities). Faraday's law and Ampère's law. Time-dependent fields $\mathbf{E}$, $\mathbf{B}$ and potentials Φ, $\mathbf{A}$. Displacement current $\mathbf{J}_{\rm D}$.
- *Electrodynamics with linear material responses.*
  Maxwell field equations in matter. Polarization current $\mathbf{J}_{\rm P}$ from electric polarization Relations between matter fields $\mathbf{P}, \mathbf{M}$ and fundamental fields $\mathbf{E}, \mathbf{B}$ expressed via fields $\mathbf{D}, \mathbf{H}$ as linear and decoupled dielectric and magnetic responses.
- *Electrodynamics with more general material responses.*
  Coupled and, in general, nonlinear constitutive relations $\mathbf{P}[\mathbf{E}, \mathbf{B}]$ and $\mathbf{M}[\mathbf{E}, \mathbf{B}]$ between matter fields and fundamental fields. Generalized nonlinear current-density response $\mathbf{J}[\mathbf{E}, \mathbf{B}]$ (later topic).

Poynting theorem:

The rate at which charge diminishes in a region of space is determined by the rate at which current flows out of that region through its surface:

$$-\frac{d}{dt}\int_V d^3x\rho = \oint_S d\mathbf{a}\cdot\mathbf{J} = \int_V d^3x\nabla\cdot\mathbf{J} \quad\Rightarrow\quad -\frac{\partial\rho}{\partial t} = \nabla\cdot\mathbf{J}.$$

The continuity equation expresses a conservation law for electric charge.



Energy conservation is equally well established. Consider a region of space with electric and magnetic fields and with ideal sources. Electromagnetic energy is carried by the electric field $\mathbf{E}(\mathbf{x},t)$ and the magnetic field $\mathbf{B}(\mathbf{x},t)$.

The rate at which electromagnetic energy in a region of space diminishes is determined (i) by the rate at which it flows out of that region through its surface and (ii) the rate at which it is converted into different kinds of energy.

Derivation of an amended continuity equation for electromagnetic energy:

– Ampère's law: $\mathbf{J} = \dfrac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$.

– Rates of energy density: $\mathbf{E}\cdot\mathbf{J} = \dfrac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\dfrac{\partial\mathbf{E}}{\partial t}$.

– Identity and Faraday's law: $\mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot\underbrace{(\nabla\times\mathbf{E})}_{-\partial\mathbf{B}/\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{B})$.

– Dot products: $\mathbf{E}\cdot\dfrac{\partial\mathbf{E}}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial t}E^2, \quad \mathbf{B}\cdot\dfrac{\partial\mathbf{B}}{\partial t} = \dfrac{1}{2}\dfrac{\partial}{\partial t}B^2$.

– Energy-density rates again:

$$\Rightarrow\ \mathbf{E}\cdot\mathbf{J} = -\nabla\cdot\underbrace{\left[\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right]}_{\mathbf{S}} - \frac{\partial}{\partial t}\underbrace{\left[\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right]}_{u}.$$

– Poynting theorem: $-\dfrac{\partial u}{\partial t} = \nabla\cdot\mathbf{S} + \mathbf{E}\cdot\mathbf{J}$.

Electromagnetic energy density:

$$u(\mathbf{x},t) = \frac{\epsilon_0}{2}E^2(\mathbf{x},t) + \frac{1}{2\mu_0}B^2(\mathbf{x},t) \quad [\mathrm{Jm^{-3}}]. \tag{14}$$

Energy current density (named Poynting vector): [lex142][lex143]

$$\mathbf{S}(\mathbf{x},t) = \frac{1}{\mu_0}\,\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t) \quad [\mathrm{Jm^{-2}s^{-1}}]. \tag{15}$$

Integral version of the Poynting theorem:

$$\underbrace{-\frac{d}{dt}\int_V d^3x\, u}_{dU_\mathrm{field}/dt} = \oint_S d\mathbf{a}\cdot\mathbf{S} + \underbrace{\int_V d^3x\, \mathbf{E}\cdot\mathbf{J}}_{dU_\mathrm{mech}/dt}, \tag{16}$$

Energy conservation: $\dfrac{dU}{dt} = \dfrac{d}{dt}\big(U_\mathrm{mech} + U_\mathrm{field}\big) = -\oint_S d\mathbf{a}\cdot\mathbf{S}.$

The right-hand side represents energy flux carried by the fields. The tacit assumption is that no energy is carried by particles through the surface S. [lex89]

In situations with $\mathbf{E}\cdot\mathbf{J} > 0$, the mobile charge carriers pick up kinetic energy. In normal conductors, that energy is continually converted into thermal energy as mobile charge carriers undergo inelastic collisions. Situations with
[lln19] $\mathbf{E}\cdot\mathbf{J} < 0$ are associated with radiation.

In the presence of dielectric and magnetic materials, the expressions for energy density, the Poynting vector, and the Poynting theorem must be modified as follows:

$$u(\mathbf{x},t) = \frac{1}{2}\big[\mathbf{E}(\mathbf{x},t)\cdot\mathbf{D}(\mathbf{x},t) + \mathbf{H}(\mathbf{x},t)\cdot\mathbf{B}(\mathbf{x},t)\big],$$

$$\mathbf{S}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t)\times\mathbf{H}(\mathbf{x},t),$$

$$-\frac{\partial u}{\partial t} = \nabla\cdot\mathbf{S} + \mathbf{E}\cdot\mathbf{J}_\mathrm{f}.$$

Not included on the right-hand side are losses associated with time-varying polarization and magnetization.

Polarizable materials are characterized by a complex function $\epsilon(\omega)$ and magnetizable materials by a complex function $\mu(\omega)$. Their imaginary parts govern the aforementioned losses.

Momentum density of electromagnetic field:

Newton's second law for charged particle in electromagnetic field:

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}.$$

Transcription to volume V with charge density ρ and current density $\mathbf{J}$:

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \int_V d^3x \big[\rho(\mathbf{x},t)\mathbf{E}(\mathbf{x},t) + \mathbf{J}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t)\big].$$

$\mathbf{P}_{\text{mech}}$ is the momentum inside V carried by the sources. The right-hand side contains a term, $-d\mathbf{P}_{\text{field}}/dt$, where $\mathbf{P}_{\text{field}}$ represents the momentum carried by the electric and magnetic fields.

Eliminate sources from integrand by using Maxwell equations:

- Use Gauss's law for electric field: $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \quad \Rightarrow \ \rho\mathbf{E} = \epsilon_0 \mathbf{E}(\nabla \cdot \mathbf{E})$.
- Use Ampère's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}$.
  $$\Rightarrow \ \mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}.$$
- Substitute into integrand (after switching factors in cross products):
  $$\Rightarrow \ \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \epsilon_0 \underbrace{\mathbf{E}(\nabla \cdot \mathbf{E})}_{a} + \epsilon_0 \underbrace{\mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t}}_{b} - \epsilon_0 c^2 \mathbf{B} \times (\nabla \times \mathbf{B}).$$
- Process terms a and b: $\quad a = \mathbf{E}(\nabla \cdot \mathbf{E}) + c^2 \mathbf{B} \underbrace{(\nabla \cdot \mathbf{B})}_{0}$.
  $$b = \frac{\partial}{\partial t}(\mathbf{B} \times \mathbf{E}) - \frac{\partial \mathbf{B}}{\partial t} \times \mathbf{E} = -\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}).$$
  $$\Rightarrow \ \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} + \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \underbrace{\epsilon_0 \big[\mathbf{E}(\nabla \cdot \mathbf{E}) + c^2\mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{E} \times (\nabla \times \mathbf{E}) - c^2 \mathbf{B} \times (\nabla \times \mathbf{B})\big]}_{d}.$$
  Note the balanced appearance of $\mathbf{E}$ and $\mathbf{B}$ in the integrand d. The factors c^2 ensures that all terms have the same units.
- Field momentum: $\mathbf{P}_{\text{field}} = \dfrac{1}{\mu_0 c^2} \displaystyle\int_V d^3x\, \mathbf{E} \times \mathbf{B} = \int_V d^3x \frac{\mathbf{S}}{c^2}$.
  Field momentum density is encoded in Poynting vector: $\mathbf{p}_{\text{field}} = \dfrac{\mathbf{S}}{c^2}$. [lex233]

– Process term d using Cartesian coordinates x_1, x_2, x_3:

$$[\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E})]_1$$
$$= E_1\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) - E_2\left(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}\right) + E_3\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}\right)$$
$$= \frac{\partial}{\partial x_1}(E_1^2) + \frac{\partial}{\partial x_2}(E_1E_2) + \frac{\partial}{\partial x_3}(E_1E_3) - \frac{1}{2}\frac{\partial}{\partial x_1}\big(E_1^2 + E_2^2 + E_3^2\big).$$

$$\Rightarrow\ [\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E})]_k = \sum_l \frac{\partial}{\partial x_l}\big(E_kE_l - \tfrac{1}{2}\mathbf{E}\cdot\mathbf{E}\,\delta_{kl}\big).$$

The same transformations pertain to the **B**-terms of integrand d.

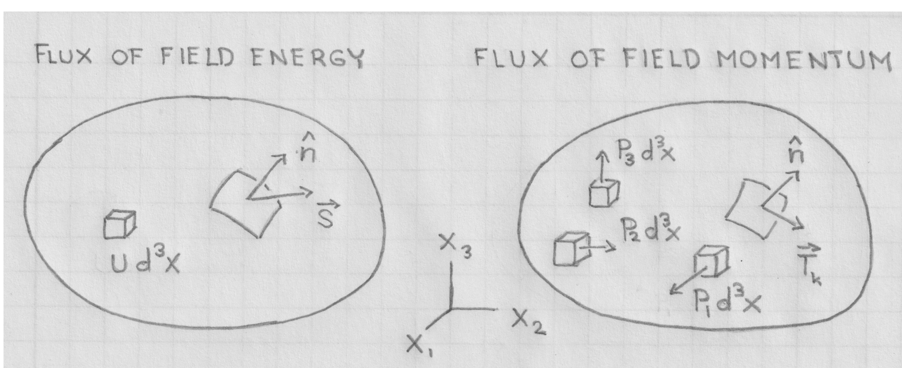
– Maxwell stress tensor (measure for momentum flux density): [lex206]

$$T_{kl} = \epsilon_0\big[E_kE_l + c^2B_kB_l - \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})\delta_{kl}\big].$$

Momentum conservation:

$$\frac{d}{dt}\big(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}}\big)_k = \int_V d^3x \sum_l \frac{\partial}{\partial x_l}T_{kl} = \oint_S da \sum_l T_{kl}n_l.$$

[lln25] The role of the momentum flux tensor will be further discussed in a relativistic context.



The direction of the flow of energy density, u (scalar quantity), is governed by the Poynting vector. It flows in the direction of $\mathbf{S} = S_1\hat{\mathbf{i}} + S_2\hat{\mathbf{j}} + S_3\hat{\mathbf{k}}$.

The direction of the flow of field momentum density $\mathbf{p} = \mathbf{S}/c^2$ (vector quantity) is governed by the Maxwell stress tensor. Each component p_kd^3x flows opposite to the direction of $\mathbf{T}_k = T_{k1}\hat{\mathbf{i}} + T_{k2}\hat{\mathbf{j}} + T_{k3}\hat{\mathbf{k}}$.

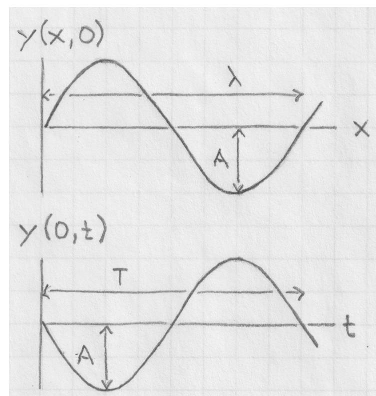
Electromagnetic waves in vacuum:

Q: What is the dynamics of the fields, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, in the absence of any sources, $\rho(\mathbf{x},t) \equiv 0$ and $\mathbf{J}(\mathbf{x},t) \equiv 0$?

A: A superposition of transverse waves of the two fields without medium, traveling at the speed of light in every reference system.

Specifications of a transverse, sinusoidal, plane wave:

- wave function: $y(x,t) = A\sin(kx - \omega t)$,
- amplitude: A,
- wave number: $k = \dfrac{2\pi}{\lambda}$,
- wavelength: λ,
- angular frequency: $\omega = \dfrac{2\pi}{T} = 2\pi f$,
- frequency: $f = \dfrac{\omega}{2\pi} = \dfrac{1}{T}$,
- period: T,
- wave speed $c = \dfrac{\lambda}{T} = \lambda f = \dfrac{\omega}{k}$,
- wave equation: $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial t^2}$.



Maxwell's equations in vacuum:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Derivation of wave equations for $\mathbf{E}$ and $\mathbf{B}$, using $c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}$$

$$\Rightarrow \nabla \underbrace{(\nabla \cdot \mathbf{E})}_{0} - \nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} \underbrace{\nabla \times \mathbf{B}}_{\mu_0\epsilon_0 \partial\mathbf{E}/\partial t} \quad \Rightarrow \nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

$$\nabla \times \mathbf{B} = \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \Rightarrow \nabla \times (\nabla \times \mathbf{B}) = \mu_0\epsilon_0 \nabla \times \frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow \nabla \underbrace{(\nabla \cdot \mathbf{B})}_{0} - \nabla^2 \mathbf{B} = -\mu_0\epsilon_0 \frac{\partial}{\partial t} \underbrace{\nabla \times \mathbf{E}}_{-\partial\mathbf{B}/\partial t} \quad \Rightarrow \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

The wave functions $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are not independent. They are coupled by Faraday's law and Ampère's law.

General plane-wave solution:

The wave equation is a *linear* PDE. The superposition principle applies. The general solution of the wave equations for $\mathbf{E}$ and $\mathbf{B}$ can be expressed as a superposition of linearly polarized plane waves of the form,

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 \sin(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi_E), \quad \mathbf{B}(\mathbf{x},t) = \mathbf{B}_0 \sin(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi_B).$$

(a) The directions of the wave vector $\mathbf{k} = k_x\,\hat{\mathbf{i}} + k_y\,\hat{\mathbf{j}} + k_z\,\hat{\mathbf{k}}$ and the amplitudes $\mathbf{E}_0, \mathbf{B}_0$ are mutually perpendicular, and $\phi_E = \phi_B$:

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\cdot\mathbf{B} = 0 \quad \Rightarrow \ \mathbf{E}_0\cdot\mathbf{k} = 0, \quad \mathbf{B}_0\cdot\mathbf{k} = 0.$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$
$$\Rightarrow \ \phi_E = \phi_B \doteq \phi, \quad \mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0, \quad \mathbf{k}\times\mathbf{B}_0 = -\mu_0\epsilon_0\omega\mathbf{E}_0.$$

The vectors $\mathbf{E}_0, \mathbf{B}_0, \mathbf{k}$ form a right-handed triad.

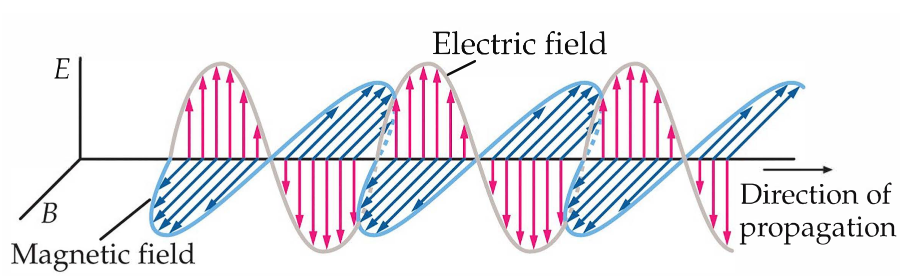
(b) The ratio $|\mathbf{E}_0|/|\mathbf{B}_0|$ is a constant and the magnitude of $\mathbf{k}$ depends on the angular frequency ω:

$$\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0, \quad \mathbf{k}\times\mathbf{B}_0 = -\mu_0\epsilon_0\omega\mathbf{E}_0$$
$$\Rightarrow \ |\mathbf{k}||\mathbf{E}_0| = \omega|\mathbf{B}_0|, \quad |\mathbf{k}||\mathbf{B}_0| = \mu_0\epsilon_0\omega|\mathbf{E}_0|$$
$$\Rightarrow \ \frac{|\mathbf{E}_0|}{|\mathbf{B}_0|} = \frac{\omega}{|\mathbf{k}|}, \quad \frac{\omega}{|\mathbf{k}|} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

(c) The Poynting vector of the plane-wave solution has the direction of the wave vector $\mathbf{k}$ and represents the energy current density:

$$\mathbf{S}(\mathbf{x},t) = \frac{1}{\mu_0}\mathbf{E}_0\times\mathbf{B}_0 \sin^2(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi) \quad [\mathrm{Jm^{-2}s^{-1}}].$$

The electric and magnetic fields oscillate (locally) in two mutually perpendicular transverse directions. [lex141]



Energy and momentum densities:

Electric and magnetic fields of linearly polarized, plane, traveling wave:

$$\mathbf{E}(x,t) = E_0 \sin(kx - \omega t)\,\hat{\mathbf{j}}, \quad \mathbf{B}(x,t) = B_0 \sin(kx - \omega t)\,\hat{\mathbf{k}}.$$

The Poynting vector,

$$\mathbf{S}(x,t) = \frac{1}{\mu_0}\mathbf{E}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t) = \frac{E_0 B_0}{\mu_0}\,\sin^2(kx - \omega t)\,\hat{\mathbf{i}} \qquad [\mathrm{W/m^2}],$$

represents the energy current density (energy per unit area and unit time transported across a surface).

Energy density profile of wave traveling at speed c in the direction of $\mathbf{S}$:

$$\begin{aligned} u(x,t) &= \frac{|\mathbf{S}(x,t)|}{c} = \frac{E_0 B_0}{\mu_0 c}\,\sin^2(kx - \omega t) \\ &= \frac{\epsilon_0}{2} E_0^2 \sin^2(kx - \omega t) + \frac{1}{2\mu_0} B_0^2 \sin^2(kx - \omega t) \qquad [\mathrm{J/m^3}]. \end{aligned}$$

Intensity (average power per unit area transported across a surface):

$$I = |\bar{\mathbf{S}}| = \frac{E_0 B_0}{2\mu_0}.$$

Momentum density carried by wave: $\dfrac{\mathbf{P}_{\mathrm{field}}}{\Delta V} = \dfrac{\mathbf{S}}{c^2} \qquad \left[\dfrac{\mathrm{kg\,m/s}}{\mathrm{m^3}}\right] = \left[\dfrac{\mathrm{W/m^2}}{(\mathrm{m/s})^2}\right].$

Average momentum density (magnitude): $\dfrac{\bar{P}_{\mathrm{field}}}{\Delta V} = \dfrac{I}{c^2}.$

Average impulse exerted by wave absorbed by flat surface: $\bar{P}_{\mathrm{field}} = \bar{F}dt.$

Radiation pressure when wave is absorbed:

$$\pi_{\mathrm{abs}} = \frac{\bar{F}}{A} = \frac{\bar{P}_{\mathrm{field}}}{Adt} = \frac{\bar{P}_{\mathrm{field}}}{Adx}\frac{dx}{dt} = \frac{\bar{P}_{\mathrm{field}}}{\Delta V}\,c = \frac{I}{c} = \frac{\bar{S}}{c} \qquad \left[\frac{\mathrm{W/m^2}}{\mathrm{m/s}}\right] = [\mathrm{N/m^2}].$$

Radiation pressure when wave is reflected: $\pi_{\mathrm{ref}} = \dfrac{\bar{F}}{A} = \dfrac{2\bar{P}_{\mathrm{field}}}{Adt} = \dfrac{2\bar{S}}{c}.$

Spherical wave solution:

The equations that govern the dynamics of the scalar and vector potentials in the absence of sources, $\rho \equiv 0$ and $\mathbf{J} \equiv 0$, have the form of wave equations, if the Lorenz gauge condition is imposed:

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}, \quad \nabla^2\mathbf{A} = \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}.$$

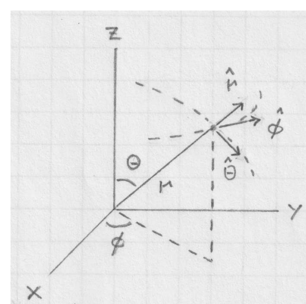
Vector potential of spherical wave moving away from a point-like source (to be identified in [lln19]):

$$\mathbf{A}(\mathbf{x},t) = \psi(r,t)\,\hat{\mathbf{k}} = A_r(r,\theta,t)\,\hat{\mathbf{r}} + A_\theta(r,\theta,t)\,\hat{\boldsymbol{\theta}},$$

$$A_r = \psi(r,t)\cos\theta, \quad A_\theta = -\psi(r,t)\sin\theta, \quad \psi(r,t) = \frac{C}{r}\,e^{\imath(kr-\omega t)}.$$

This ansatz satisfies the wave equation. The direction of $\mathbf{A}$ determines the polarization of the radial wave (directions of $\mathbf{E}$ and $\mathbf{B}$). Derivations of $\mathbf{E}$ and $\mathbf{B}$ from $\mathbf{A}$: [lex84][lex85]

$$\begin{aligned}
\mathbf{B}(\mathbf{x},t) &= \nabla\times\mathbf{A}(\mathbf{x},t) = B_\phi(r,t)\,\hat{\boldsymbol{\phi}}, \\
\mathbf{E}(\mathbf{x},t) &= -\frac{c^2}{\imath\omega}\nabla\times\mathbf{B}(\mathbf{x},t) = E_r(r,t)\,\hat{\mathbf{r}} + E_\theta(r,t)\,\hat{\boldsymbol{\theta}}, \\
B_\phi(r,t) &= \left(-\imath k + \frac{1}{r}\right)\sin\theta\,\psi(r,t), \\
E_r(r,t) &= \frac{2c^2}{\imath\omega}\left(\frac{\imath k}{r} - \frac{1}{r^2}\right)\cos\theta\,\psi(r,t), \\
E_\theta(r,t) &= \frac{c^2}{\imath\omega}\left(k^2 + \frac{\imath k}{r} - \frac{1}{r^2}\right)\sin\theta\,\psi(r,t).
\end{aligned}$$



In the *radiation zone* (at large r), the most slowly decaying (asymptotic) terms are dominant:

$$\begin{aligned}
\mathbf{B}_{\rm as}(\mathbf{x},t) &= -\imath k\sin\theta\,\psi(r,t)\,\hat{\boldsymbol{\phi}} = \frac{Ck}{r}\sin\theta\,\sin(kr-\omega t)\,\hat{\boldsymbol{\phi}} \quad (\text{east} - \text{west}), \\
\mathbf{E}_{\rm as}(\mathbf{x},t) &= -\imath\omega\sin\theta\,\psi(r,t)\,\hat{\boldsymbol{\theta}}, = \frac{C\omega}{r}\sin\theta\,\sin(kr-\omega t)\,\hat{\boldsymbol{\theta}} \quad (\text{south} - \text{north}).
\end{aligned}$$

Poynting vector in radiation zone:

$$\mathbf{S}_{\rm as}(\mathbf{x},t) = \frac{1}{\mu_0}\mathbf{E}_{\rm as}(\mathbf{x},t)\times\mathbf{B}_{\rm as}(\mathbf{x},t) = \frac{C^2\omega k}{\mu_0 r^2}\,\sin^2\theta\,\sin^2(kr-\omega t)\,\hat{\mathbf{r}}.$$

The radiation intensity depends on r and θ. The total intensity is a constant.