Electrodynamics I [lln14]

In electrostatics, the sources of electric field are free or bound electric charges at rest. In magnetostatics, the sources of magnetic field are time-independent configurations of conduction currents and bound currents.

Electrodynamics analyzes situations with time-dependent electric fields $\mathbf{E}(\mathbf{x}, t)$ and magnetic fields $\mathbf{B}(\mathbf{x}, t)$.

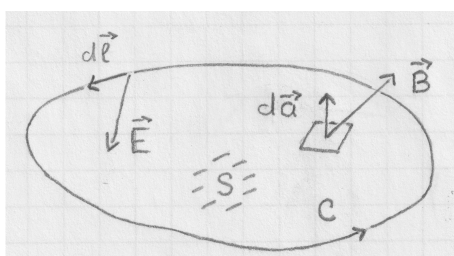
[lln15] An additional source of electric field is associated with magnetic induction (discussed here). An additional source of magnetic field is associated with displacement currents (discussed later).

In a nutshell: time-varying magnetic fields cause electric fields and vice versa.

Faraday's law of electromagnetic induction:

Consider the perimeter loop C of an open surface S shown. It can be a physical loop (e.g. a wire) or a loop of your imagination.

- The loop C is a closed curve in space with a sense of traversal (indicated by arrow). The loop must not intersect itself, but it can be twisted.
- The open surface S is surrounded by the perimeter loop C. The surface of a given loop is not unique, in general. A planar loop has a unique flat surface.



- Line element $d\mathbf{l}$: vector with SI unit [m] representing an infinitesimal segment of the perimeter loop C in the direction of traversal.
- Area element $d\mathbf{a}$: vector with SI units [m^2]. The magnitude of $d\mathbf{a}$ is the area of a surface element on S. The direction of $d\mathbf{a}$ is normal to S the surface in the direction consistent with the traversal of C.[1]

[1]The consistency is dictated by the right-hand rule: Curve the fingers of your right hand in the direction of C at any location of the loop. Then your thumb marks the side of S to which $d\mathbf{a}$ points.

– Magnetic flux: $\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a}$ $[\mathrm{Tm}^2 = \mathrm{Wb}]$ (Weber).

– Time-dependent electric fields are not, in general, irrotational.

– Electromotive force (EMF): $\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l}$ [V].

– Faraday's law (integral version):

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} = -\frac{d\Phi_B}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \quad [\mathrm{V} = \mathrm{Wb/s}].$$

[gmd1-B] – Stokes' theorem: $\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{a}$.

– Faraday's law (differential version): $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

General comments:

– Calculating a flux quantity through an open surface always involves a choice of direction:

▷ Electric current I is the flux of the current density $\mathbf{J}$, typically through an (open) cross sectional surface. $\mathbf{J}$ is a vector field with a definite direction. I (not a vector) assumes a direction associated with the area vector used in the construction of flux.

▷ The flux Φ_B of a magnetic field $\mathbf{B}$ through a loop assumes a direction in an analogous way.

▷ In both instances, the direction of the flux quantity is tied to the direction of the chosen area vector via the right-hand rule. The direction of the field is determined by the physical situation.

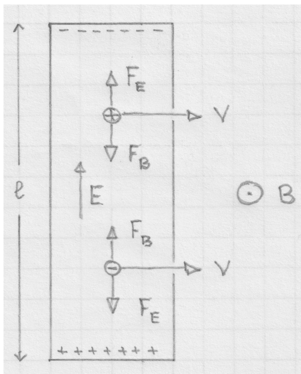
– Switching the sense of traversal on C implies switching the direction of the area vector on S. The integrity of Faraday's law is not affected by the choice made.

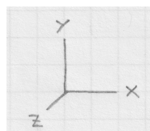
Lenz's rule (statement of negative feedback): The magnetic field generated by induced currents opposes the change in magnetic flux which causes it.

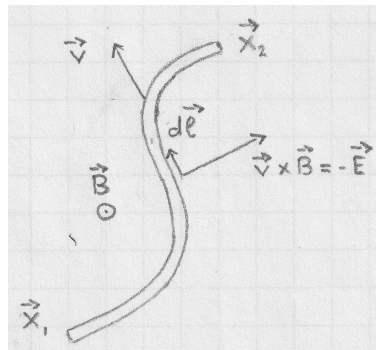
Note that Faraday's law can stand on its own. Lenz's rule is merely a convenient tool for the determination of the direction of induced currents.

Motional EMF:

A conducting rod moves laterally in uniform magnetic field.







- Rod of length l oriented in y-direction, moving in x-direction.
- Velocity: $\mathbf{v} = v\,\hat{\mathbf{i}}$.
- Magnetic field: $\mathbf{B} = B\,\hat{\mathbf{k}}$.
- Magnetic force on mobile charge carriers: $\mathbf{F}_B = q\,\mathbf{v} \times \mathbf{B} = -qvB\,\hat{\mathbf{j}}$.
- Magnetic force causes vertical drift of mobile charge carriers.
- Charge separation causes buildup of electric field: $\mathbf{E} = E\,\hat{\mathbf{j}}$.
- Electric force $\mathbf{F}_E = q\,\mathbf{E}$ counteracts magnetic force $\mathbf{F}_B$.
- Drift ceases when forces are in balance: $\mathbf{F}_E = -\mathbf{F}_B$.
- Electric field built up: $\mathbf{E} = -\mathbf{v} \times \mathbf{B} = vB\,\hat{\mathbf{j}}$.
- Motional EMF for this case: $\mathcal{E} = -\mathbf{E} \cdot \mathbf{l} = vBl$.

Motional EMF for an open wire is akin to an electrostatic potential: [lex81]
[lex137][lex138]

$$\mathcal{E} \doteq -\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E} \cdot d\mathbf{l} = \int_{\mathbf{x}_1}^{\mathbf{x}_2} d\mathbf{l} \cdot \mathbf{v} \times \mathbf{B}.$$

Electric and magnetic forces are balanced for mobile charge carriers in each segment $d\mathbf{l}$ of the wire.

The principle of relativity in the context of motional EMF will be a topic in
[lln25] a later module.

Consistency of motional EMF with Faraday's law:

Conducting loop C (wire) moving in static magnetic field of arbitrary shape.

– Motional EMF: $\mathcal{E} = \oint_C d\mathbf{l} \cdot \mathbf{v} \times \mathbf{B} = \oint_C \mathbf{B} \cdot d\mathbf{l} \times \mathbf{v}$.

– Displacement of loop: $d\mathbf{x} = \mathbf{v}dt$.

– Loops C_t at time t and C_{t+dt} at $t + dt$ define a looping ribbon.

– Area element on ribbon: $d\mathbf{a} = d\mathbf{l} \times d\mathbf{x}$.

– The loop direction $d\mathbf{l}$ is chosen such that $d\mathbf{a}$ points outward.

– Magnetic flux through ribbon: $d\Phi_B^{(\mathrm{rib})} = \oint_C \mathbf{B} \cdot \underbrace{d\mathbf{l} \times d\mathbf{x}}_{d\mathbf{a}} = \mathcal{E} dt$.

– Construct closed surface S_c from ribbon and two open surfaces S_t and S_{t+dt} inside perimeters C_t and C_{t+dt}, respectively.

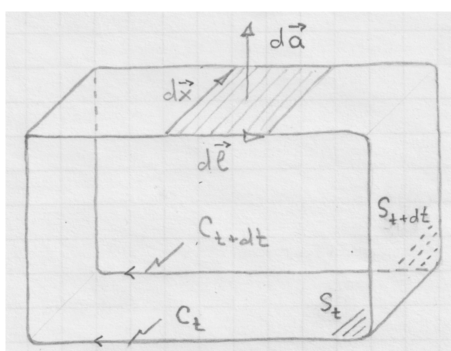
– Use Gauss's law for the magnetic field:

$$\Rightarrow \quad \oint_{S_\mathrm{c}} d\mathbf{a} \cdot \mathbf{B} = \mathcal{E} dt + \int_{S_{t+dt}} d\mathbf{a} \cdot \mathbf{B} - \int_{S_t} d\mathbf{a} \cdot \mathbf{B} = 0.$$

– In the next step, we divide by dt and take the limit $dt \to 0$.

– Of the flux through a closed surface remains the difference of flux through two open surfaces.

– Faraday's law recovered: $\mathcal{E} = -\frac{d}{dt} \int_{S_t} d\mathbf{a} \cdot \mathbf{B} = -\frac{d\Phi_B}{dt}$.



Faraday disk generator:

Conducting disk spinning in plane perpendicular to uniform magnetic field. Stationary wire connects axle with rim by sliding contacts.

Cylindrical symmetry is broken by sliding contact. It is (effectively) restored when the rim of the disk is made of a layer with significantly higher conductivity. Simplifying feature: current density inside disk is radial.

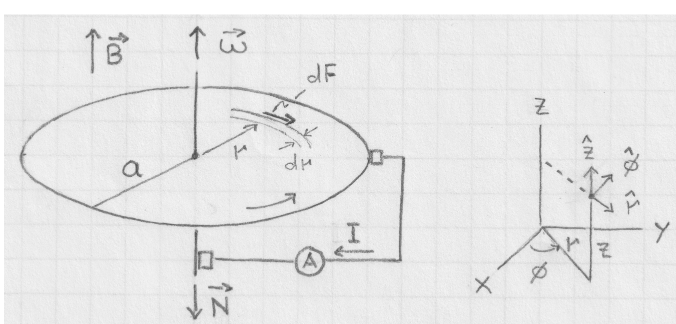
- Magnetic field: $\mathbf{B} = B\,\hat{\mathbf{z}}$ (uniform).
- Radius of the disk: a.
- Angular velocity: $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$.
- Motional EMF: $\mathcal{E} = \int_0^a dr\, r\omega B = \frac{1}{2}\omega B a^2$.
- Disk with sliding contacts is a resistor (with resistance R).
- Current between sliding contacts: $I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$.
- Current in disk is radially outward: $d\mathcal{E} = \mathbf{B}\cdot d\mathbf{r}\times\mathbf{v} > 0$.
- Current density inside disk: $\mathbf{K} = K(r)\,\hat{\mathbf{r}}$ (per unit length) [A/m].
- Radial current inside disk: $I = 2\pi r K(r)$ (independent of r).
- Magnetic attenuation force acting on ring of radius r and width dr:

  $$d\mathbf{F} = (2\pi r dr)\mathbf{K}\times\mathbf{B} = -2\pi r dr K(r) B\,\hat{\boldsymbol{\phi}} \quad \text{(tangential to circle).}$$
- Attenuating torque acting on disk: [lex74][lex75]

$$\mathbf{N} = \int_{\text{disk}} \mathbf{r}\times d\mathbf{F} = \int_0^a dr\,(2\pi r)\mathbf{r}\times(\mathbf{K}\times\mathbf{B}),$$

$$\mathbf{r}\times(\mathbf{K}\times\mathbf{B}) = \mathbf{K}\underbrace{(\mathbf{r}\cdot\mathbf{B})}_{0} - \mathbf{B}(\mathbf{r}\cdot\mathbf{K}) = -rK(r)\,\mathbf{B}.$$

$$\Rightarrow\ \mathbf{N} = -\mathbf{B}\int_0^a dr(2\pi r) r K(r) = -\mathbf{B} I \int_0^a dr\, r = -\frac{1}{2} I B a^2\,\hat{\mathbf{z}}.$$



Eddy currents and magnetic attenuation:

Objects made of conducting materials experience *eddy currents* when moved into or out of regions of magnetic fields or, more generally, through a region of inhomogeneous magnetic field.

Eddy currents inside objects that are extended in more than one dimension are, in general, of complex shape and hard to quantify. A quantitative analysis is much simplified if the object is effectively one-dimensional, e.g. a loop of wire or a network of wires. [lex76][lex202]

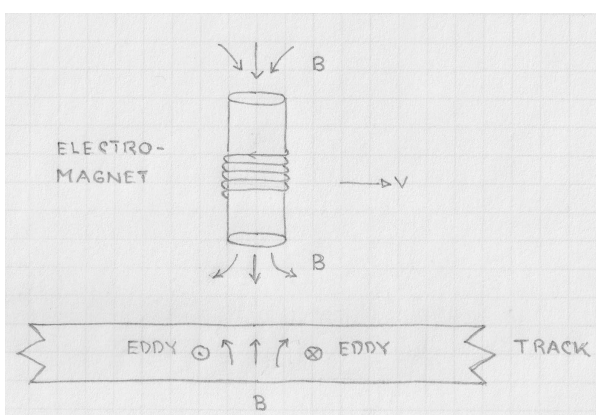
Magnetic braking is a means for slowing down moving objects as an alternative to kinetic friction:

- It converts kinetic energy into electromagnetic energy by induction.
- It dissipates electromagnetic energy into heat via eddy currents.
- The eddy currents thus produce a drag force without sliding contact.
- The wear and tear associated with frictional forces is avoided.

When a narrow source of magnetic field on a train moves along the track, the induced eddy current in the track generates a magnetic field that opposes the generating magnetic field in direction.

The direction of the induced magnetic field is consistent with Lenz's rule.

The induced magnetic field also provides a repulsive force between train and track (in the manner of the repulsive force between magnetic dipoles facing each other). This principle is exploited in *magnetic levitation* of high-speed trains.



Alternating-current generator:

Alternating-current (ac) circuits are driven by power sources that deliver an alternating EMF (via electrical outlets).

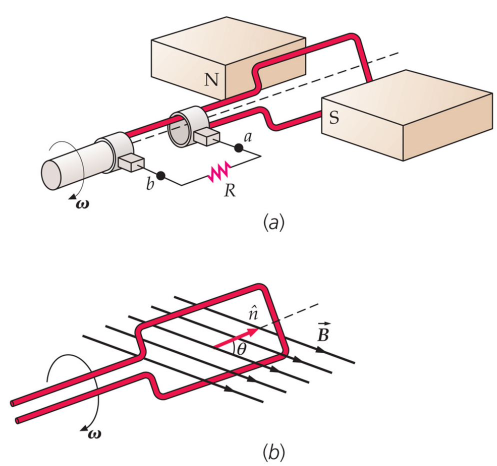
Principle of ac power generation: a mechanically powered turbine forces a conducting loop to rotate in a static magnetic field.

The steady rotation of a loop with angular velocity ω changes the angle between the area vector $\mathbf{a}$ and the magnetic field $\mathbf{B}$ continually.

In consequence, the magnetic flux Φ_B through the loop changes periodically. The rate of that change is also periodic.

- Area of conducting loop: a.
- Number of turns: N.
- Area vector: $\mathbf{a} = a\hat{\mathbf{n}}$.
- Magnetic field: $\mathbf{B}$.
- Angular velocity of rotation: ω.
- Angle between vectors $\mathbf{a}$ and $\mathbf{B}$: $\theta = \omega t$.
- Magnetic flux: $\Phi_B = N\mathbf{a} \cdot \mathbf{B} = NaB\cos(\omega t)$.
- Angular frequency of magnetic flux and induced EMF: ω.
- Induced EMF: $\mathcal{E} = -\frac{d\Phi_B}{dt} = \underbrace{NaB\omega}_{\mathcal{E}_{max}}\sin(\omega t)$.

The EMF amplitude, $\mathcal{E}_{max}$, can be made higher by increasing any one or several of four factors



(a) (b)

Cyclotron:

One of the earliest particle accelerator designs.

– Low-energy protons (mass m, charge e) are injected at S.

– Path of proton is bent by magnetic field B.

– Alternating EMF between Dee_1 and Dee_2: $\mathcal{E}(t) = \mathcal{E}_{max}\sin(\omega t)$.

– Adjustable angular frequency: ω.

– Cyclotron period: $T = \dfrac{2\pi m}{eB}$.

– One-time synchronization: $\dfrac{2\pi}{\omega} = T$ (no adjustments necessary).

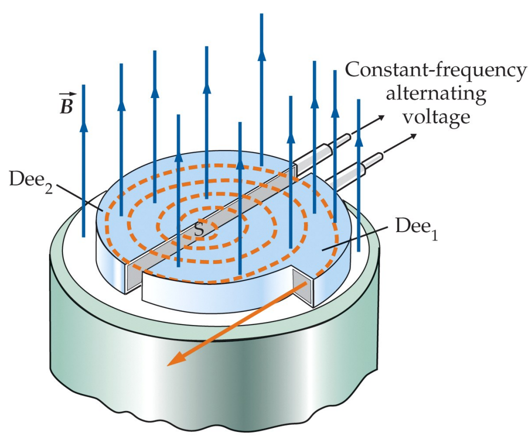
– Energy gain per half cycle of protons: $\Delta K \simeq e\mathcal{E}_{max}$.

– Path spirals out as velocity of proton increases.

– Radius grows linearly with velocity: $r = \dfrac{mv}{eB}$.

– Key attribute: period T is independent of v or r.

– High-energy protons exit at perimeter of B-field region.



This design only works for non-relativistic speeds: $v \ll c$. For particles with relativistic speeds, $v \lesssim c$, the period T depends on v.

The angular frequency ω of the EMF $\mathcal{E}(t)$ must be adjusted as the particles pick up kinetic energy. This more advanced design is named *synchrotron*.

Betatron:

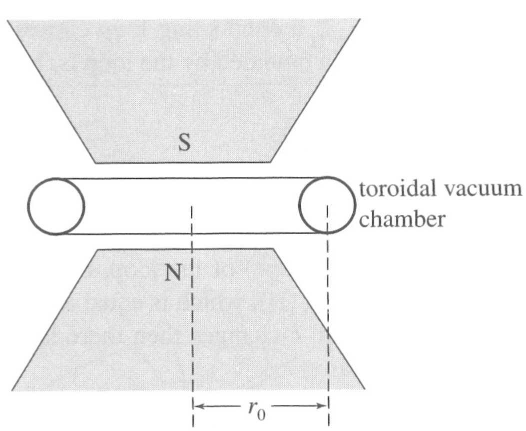
The betatron, an electron accelerator, has a design in which the magnetic field plays a dual role: (i) it keeps the electrons on a circular path by providing the centripetal force; (ii) it accelerates the electrons via induced EMF.

- Toroidal vacuum chamber of radius r_0 with electrons circling in a magnetic field $B(r,t)$ directed $\uparrow$ in the gap of an electromagnet.
- Condition for keeping the orbital radius constant when the particles pick up momentum:

$$r_0 = \frac{mv}{eB} = \frac{p(t)}{eB(r_0,t)}, \quad \Rightarrow p(t) = er_0 B(r_0,t) \quad \Rightarrow \frac{dp}{dt} = er_0 \frac{d}{dt} B(r_0,t).$$

- Average magnetic field inside orbit: $\bar{B}(t) = \frac{1}{\pi r_0^2} \int_0^{r_0} dr(2\pi r)B(r,t)$.
- Magnetic flux inside path of electrons: $\Phi_B(t) = \pi r_0^2 \bar{B}(t)$.
- Magnetic induction at radius r_0: $\mathcal{E} = 2\pi r_0 E = \frac{d\Phi_B}{dt} = \pi r_0^2 \frac{d\bar{B}(t)}{dt}$.
- Induced electric field (tangential to orbit): $E = \frac{r_0}{2} \frac{d\bar{B}(t)}{dt}$.
- Rate of momentum gain of electrons: $\frac{dp}{dt} = eE = \frac{er_0}{2} \frac{d\bar{B}(t)}{dt}$.
- Keeping the orbital radius constant in the face of a gradual momentum gain requires that local magnetic field at radius r_0 increases synchronously with the average magnetic field:

$$B(r_0,t) = \frac{1}{2}\bar{B}(t).$$



Inductance:

The phenomenon of magnetic induction as governed by Faraday's law, manifests itself in a circuit device named *inductor*.

The *solenoid* is the prototypical inductor. A long wire is tightly wound into a cylinder, Each turn of wire is, effectively, a loop.

A current sent through the wire generates a magnetic field around it. This produces a magnetic flux through each turn. A time-dependent current causes a time-dependent magnetic flux, which induces an EMF.

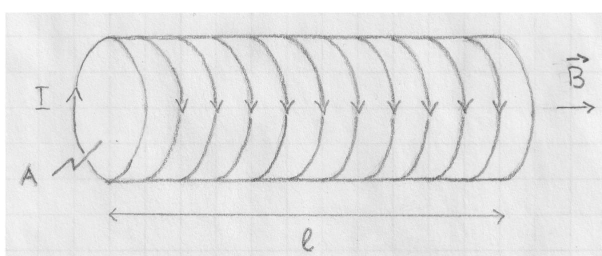
The relevant device property is the *inductance* L,

$$L \doteq \frac{N\Phi_B}{I} \quad [\mathrm{Wb/A}] = [\mathrm{H}] \quad (\mathrm{Henry}),$$

where Φ_B is the magnetic flux through each of the N turns of wire.

Inductance of a solenoid:

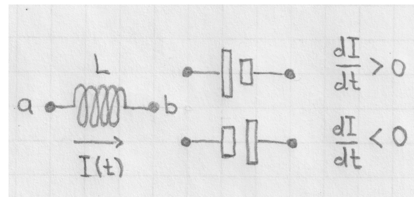
- A: cross-sectional area
- ℓ: length
- n: number of turns per unit length
- $N = n\ell$: total number of turns
- $B = \mu_0 n I$: magnetic field inside solenoid [lex65]
- $\Phi_B = BA$: magnetic flux through each turn
- Inductance of solenoid: $L \doteq \dfrac{N\Phi_B}{I} = \mu_0 n^2 A\ell$



A solenoid bent into a torus is named *toroid*. Calculating the inductance of a toroid starts from the same definition and arrives at a different result. [lex77]

Self-induction:

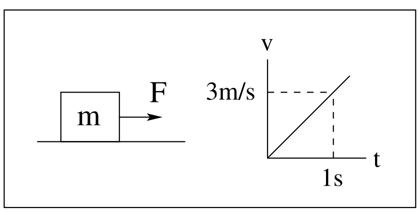
A time-dependent current $I(t)$ flowing through an inductor with inductance L induces an EMF $\mathcal{E}(t)$ in the same device:

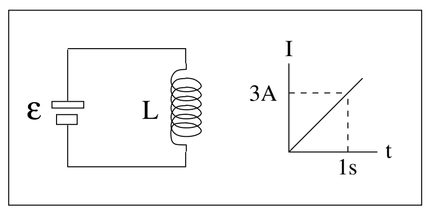


– Faraday's law: $\mathcal{E} = -\frac{d}{dt}(N\Phi_B)$.
– Inductance: $L = \frac{N\Phi_B}{I}$.
– Equivalent voltage: $V \doteq \Phi_b - \Phi_a = \mathcal{E}$.
– Self-induced EMF: $\mathcal{E} = -L\frac{dI}{dt}$.

Lenz's negative-feedback rule in action: The induced EMF acts such as to oppose the change in current that causes it.

Inductors make electric currents sluggish, which is illustrated by a mechanical analogy. Just as mass is a measure of mechanical inertia, inductance is a measure of electromagnetic inertia.





Mechanical system: A block of mass $m = 2$kg is accelerated from rest by a constant force $F = 6$N on a frictionless surface.

– Newton's second law: $F - m\frac{dv}{dt} = 0$.
– Rate of velocity increase: $\frac{dv}{dt} = \frac{F}{m} = 3\text{m/s}^2$.

Electromagnetic system: An EMF source delivering voltage $\mathcal{E} = 6$V is connected to a circuit with an inductor of inductance $L = 2$H, building up a current I from zero initial value.

– Loop rule: $\mathcal{E} - L\frac{dI}{dt} = 0$.
– Rate of current increase: $\frac{dI}{dt} = \frac{\mathcal{E}}{L} = 3\text{A/s}$.

Energy stored in inductor:

Establishing a current in the inductor requires work. The work done is equal to the potential energy stored in the device.

– Current through inductor: I (increasing from zero).

– Voltage induced across inductor: $|\mathcal{E}| = L\frac{dI}{dt}$.

– Power absorbed by inductor: $P = |\mathcal{E}|I$.

– Increment of potential energy: $dU = Pdt = LIdI$.

– Energy stored in inductor: $U = L\int_0^I IdI = \frac{1}{2}LI^2$.

– Inductance of solenoid: $L = \mu_0 n^2 A\ell$.

– Magnetic field inside solenoid: $B = \mu_0 nI$.

– Energy stored in solenoid : $U = \frac{1}{2\mu_0}B^2(A\ell)$.

– Volume of solenoid interior: $A\ell$.

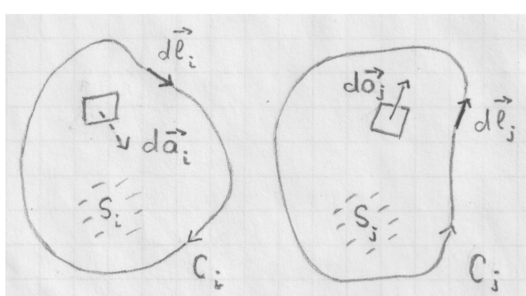
– Energy density of magnetic field: $u_B = \frac{U}{A\ell} = \frac{1}{2\mu_0}B^2$.

General energy-density expression of electric and magnetic fields in vacuum:

$$u = u_E + u_B = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \quad [\mathrm{J/m^3}].$$

Mutual induction:

Consider a configuration of unconnected loops of wire, $C_1, C_2, \ldots$, with currents $I_1, I_2, \ldots$ flowing through them, all in the same region of space.



Each loop generates a magnetic field everywhere in that region. The field strength is proportional to the current in the loop.

The magnetic flux $\Phi_B^{(i)}$ through any particular loop C_i thus depends linearly on the current I_i in the same loop and the currents I_j in all other loops:

$$\Phi_B^{(i)} = L_i I_i + \sum_{j \neq i} M_{ij} I_j \quad : \ i = 1, 2, \ldots$$

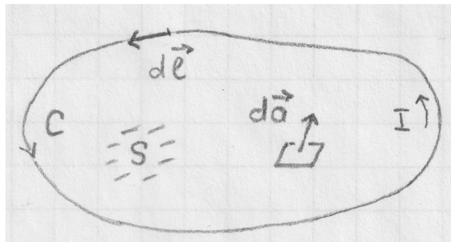
- Magnetic flux through loop C_i caused by current I_j named $\Phi_B^{(ij)}$.
- Self-inductance: $L_i \doteq \dfrac{\Phi_B^{(ii)}}{I_i}$.
- Mutual inductance: $M_{ij} \doteq \dfrac{\Phi_B^{(ij)}}{I_j} \quad : \ j \neq i$.
- Induced EMF: $\mathcal{E}_i = -L_i \dfrac{dI_i}{dt} - \sum_{j \neq i} M_{ij} \dfrac{dI_j}{dt}$.

Mutual-inductance tensor is symmetric: $M_{ij} = M_{ji}$.

- Magnetic field generated by current I_j anywhere: $\mathbf{B}_j = \nabla \times \mathbf{A}_j$.
- Loop C_i surrounds surface S_i.
- Magnetic flux through S_i caused by I_j: $\Phi_B^{(ij)} = \displaystyle\int_{S_i} d\mathbf{a}_i \cdot \mathbf{B}_j = M_{ij} I_j$.
- Stokes's theorem: $\displaystyle\int_{S_i} d\mathbf{a}_i \cdot \mathbf{B}_j(\mathbf{x}_i) = \oint_{C_i} d\mathbf{l}_i \cdot \mathbf{A}_j(\mathbf{x}_i)$.
- Mutual inductance: $M_{ij} = \dfrac{1}{I_j} \displaystyle\oint_{C_i} d\mathbf{l}_i \cdot \mathbf{A}_j(\mathbf{x}_i)$.
- Vector potential of current I_j: $\mathbf{A}_j(\mathbf{x}_i) = \dfrac{\mu_0}{4\pi} \displaystyle\oint_{C_j} \frac{I_j d\mathbf{l}_j}{|\mathbf{x}_i - \mathbf{x}_j|}$.
- Neumann equation: $M_{ij} = \dfrac{\mu_0}{4\pi} \displaystyle\oint_{C_i} \oint_{C_j} \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{|\mathbf{x}_i - \mathbf{x}_j|}$.
- Note symmetry with respect to permutations of indices.
- Convention: directions of loop integration can be chosen such that all mutual inductances M_{ij} come out positive. [lex86]

Magnetic field energy density:

Establish a general expression for the energy content of a magnetic field in a region of space. We begin with a loop of wire carrying a current I and then generalize to a region of free-current density $\mathbf{J}$.



- Loop C surrounds open surface S.
- Self-inductance of loop: L.
- Energy stored with current I running: $U = \frac{1}{2}LI^2$.
- Magnetic flux through surface S: $\Phi_B = \int_S d\mathbf{a}\cdot\mathbf{B} = LI$.
- Vector potential $\mathbf{A}$ and magnetic B-field: $\mathbf{B} = \nabla\times\mathbf{A}$.
- Stokes' theorem: $\int_S d\mathbf{a}\cdot\mathbf{B} = \int_S d\mathbf{a}\cdot\nabla\times\mathbf{A} = \oint_C d\mathbf{l}\cdot\mathbf{A}$.
- Energy expression transformed: $U = \frac{1}{2}I(LI) = \frac{1}{2}I\oint_C d\mathbf{l}\cdot\mathbf{A}$.
- Generalization to region with current density: $Id\mathbf{l} = \mathbf{J}_\mathrm{f}d^3x$.
- Energy expression transformed: $U = \frac{1}{2}\int d^3x\,\mathbf{J}_\mathrm{f}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})$.

[lln13] – Ampère's law for H-field: $\nabla\times\mathbf{H} = \mathbf{J}_\mathrm{f}$.

- Gauss's theorem implies:[2] $\int d^3x\,\nabla\cdot[\mathbf{H}(\mathbf{x})\times\mathbf{A}(\mathbf{x})] = 0$.

[gmd1-A] – Identity: $\underbrace{(\nabla\times\mathbf{H})}_{\mathbf{J}_\mathrm{f}}\cdot\mathbf{A} = \underbrace{\nabla\cdot(\mathbf{H}\times\mathbf{A})}_{\to 0} + \mathbf{H}\cdot\underbrace{(\nabla\times\mathbf{A})}_{\mathbf{B}}$.

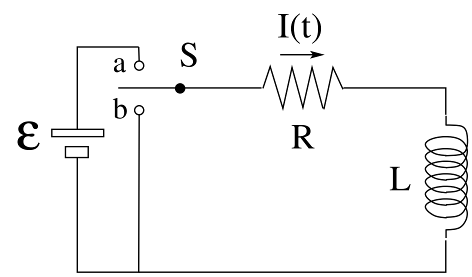
- Energy expression transformed: $U = \frac{1}{2}\int d^3x\,\mathbf{H}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})$.
- Magnetic-field energy density: $u(\mathbf{x}) = \frac{1}{2}\mathbf{H}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x})$.

[2] If the current density extends over a finite region, then $\mathbf{H}\to 0$ at $|\mathbf{x}|\to\infty$.

RL circuits:

One-loop circuit with battery, resistor, and inductor.

Specifications: EMF $\mathcal{E}$, resistance R, inductance L.



Time-dependent quantities:

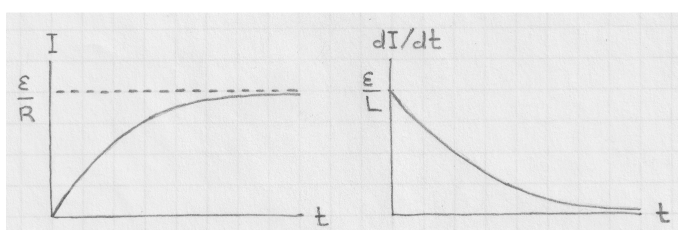
- current through loop: $I(t)$,
- rate of change of current: $\dfrac{dI}{dt}$,
- voltage across resistor: $V_R(t) = RI(t)$,
- voltage across inductor: $V_L(t) = L\,\dfrac{dI}{dt}$.

(i) *Current buildup* (battery connected):

- Loop rule: $\mathcal{E} - I(t)R - L\,\dfrac{dI}{dt} = 0$.
- ODE solved by separation of variables:

$$\frac{dI}{dt} = \frac{\mathcal{E}/R - I}{L/R} \quad \Rightarrow \quad \int_0^I \frac{dI}{\mathcal{E}/R - I} = \int_0^t \frac{dt}{L/R} \quad \Rightarrow -\ln\left(\frac{\mathcal{E}/R - I}{\mathcal{E}/R}\right) = \frac{t}{L/R}.$$

- Current: $I(t) = \dfrac{\mathcal{E}}{R}\Big[1 - e^{-Rt/L}\Big]$.
- Rate of current change: $\dfrac{dI}{dt} = \dfrac{\mathcal{E}}{L}\,e^{-Rt/L} > 0$.



- Rates of energy transfer: $I(t)\mathcal{E} - I^2(t)R - LI(t)\dfrac{dI}{dt} = 0$,

  $I\mathcal{E} > 0$: rate at which battery delivers energy,

  $IV_R = I^2R > 0$: rate at which resistor dissipates energy,

  $IV_L = LI\,\dfrac{dI}{dt} > 0$: rate at which inductor stores energy.

– Buildup characteristics:

$\dfrac{\mathcal{E}}{L} = \left.\dfrac{dI}{dt}\right|_{t=0}$: initial rate at which current increases,

$\dfrac{\mathcal{E}}{R} = \lim_{t\to\infty} I(t)$: final (steady) current,

$\dfrac{L}{R}$: time it takes to build up ∼63% of the current.

(ii) *Current shutdown* (battery disconnected):

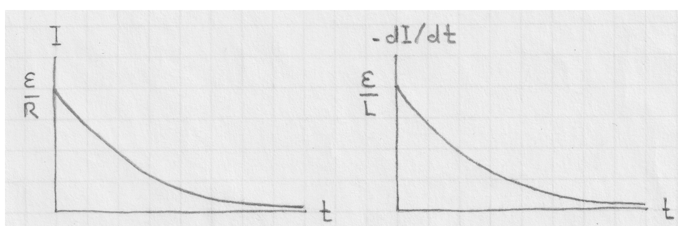
– Loop rule: $I(t)R + L\dfrac{dI}{dt} = 0.$

– ODE solved by separation of variables:

$$\frac{dI}{dt} = -\frac{R}{L}I \quad \Rightarrow \quad \int_{\mathcal{E}/R}^{I} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt \quad \Rightarrow \quad \ln\left(\frac{I}{\mathcal{E}/R}\right) = -\frac{R}{L}t.$$

– Current: $I(t) = \dfrac{\mathcal{E}}{R}e^{-Rt/L}.$

– Rate of current change: $\dfrac{dI}{dt} = -\dfrac{\mathcal{E}}{L}e^{-Rt/L} < 0.$



– Rates of energy transfer: $I^2(t)R + LI(t)\dfrac{dI}{dt} = 0,$

$IV_R = I^2R > 0$: rate at which resistor dissipates energy,

$IV_L = LI\dfrac{dI}{dt} < 0$: rate at which inductor releases energy.

– Shutdown characteristics:

$\dfrac{\mathcal{E}}{L} = -\left.\dfrac{dI}{dt}\right|_{t=0}$: initial rate at which current decreases,

$\dfrac{\mathcal{E}}{R} = I(0)$: total current to be shut down,

$\dfrac{L}{R}$: time it takes to shut down ∼63% of the current.

- Energy stored on inductor initially: $U = \frac{1}{2}L\left(\frac{\mathcal{E}}{R}\right)^2$.
- The energy dissipated in the resistor is equal to the energy released by the inductor:

$$\int_0^\infty dt\, I^2(t)R = \frac{\mathcal{E}^2}{R}\int_0^\infty dt\, e^{-2Rt/L} = \frac{1}{2}L\left(\frac{\mathcal{E}}{R}\right)^2.$$

Key points to remember in the analysis of RL circuits:

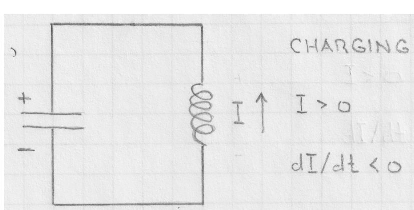
- The voltage across an inductor with a steady current is zero. The inductor with a steady current is an invisible device.
- The current through an inductor can only change gradually. Any abrupt change would require a source of infinite voltage.

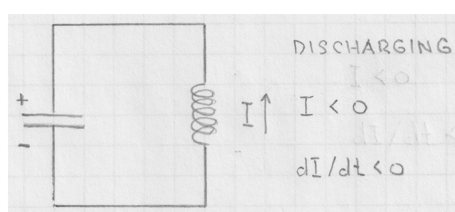
Electromagnetic oscillator (LC circuit):

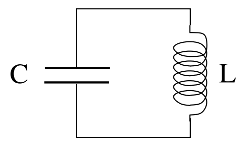
One-loop circuit with capacitor and inductor.

Specifications: capacitance C, inductance L.

- loop rule: $\frac{Q}{C} + L\frac{dI}{dt} = 0,\ I = \frac{dQ}{dt}$
- equation of motion: $\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$
- charge on capacitor: $Q(t) = Q_{max}\cos(\omega t)$
- current through inductor: $I(t) = -\omega Q_{max}\sin(\omega t)$
- angular frequency: $\omega = \frac{1}{\sqrt{LC}}$
- magnetic energy: $U_B = \frac{1}{2}LI^2$ (stored on inductor)
- electric energy: $U_E = \frac{Q^2}{2C}$ (stored on capacitor)
- total energy: $E = U_B + U_E = \text{const.}$





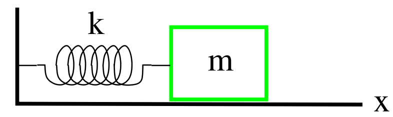


Electromagnetic oscillations:

charge: $Q(t) = A\cos(\omega t)$,

current: $I(t) = -A\omega\sin(\omega t)$,

period: $\tau = \dfrac{2\pi}{\omega}, \quad \omega = \dfrac{1}{\sqrt{LC}}$.

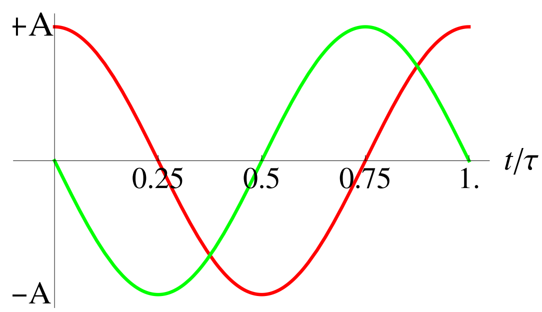


Mechanical oscillations:

position: $x(t) = A\cos(\omega t)$,

velocity: $v(t) = -A\omega\sin(\omega t)$,

period: $\tau = \dfrac{2\pi}{\omega}, \quad \omega = \sqrt{\dfrac{k}{m}}$.



Electromagnetic oscillations:

electric energy: $U_E(t) = \dfrac{1}{2C}Q^2(t)$,

magnetic energy: $U_B(t) = \dfrac{1}{2}LI^2(t)$,

total energy: $E = U_E(t) + U_B(t)$.

Mechanical oscillations:

potential energy: $U(t) = \dfrac{1}{2}kx^2(t)$,

kinetic energy: $K(t) = \dfrac{1}{2}mv^2(t)$,

total energy: $E = U(t) + K(t)$.

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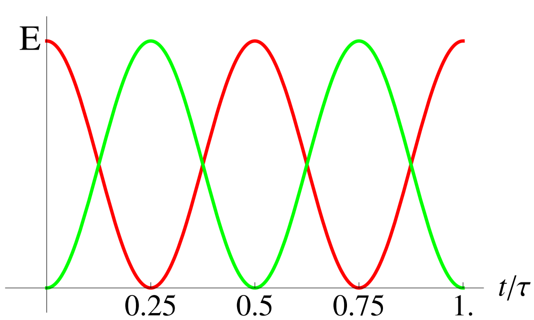


Electromagnetic oscillator with two modes:

The LC circuit shown has two inductors and three capacitors. It can oscillate in two different modes, depending on initial conditions.

In mode #1, the capacitor in the middle does not participate. It remains uncharged.

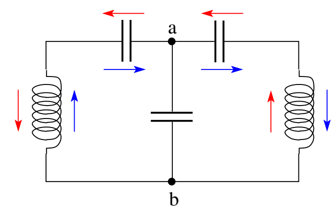
In mode #2, there are equal currents in the left and right branches. The current thus doubles in the middle branch.

All currents change direction after half a period as indicated by color code.

The angular frequency of mode #2 is higher that of mode #1. When the LC circuit is launched from an arbitrary initial state, both modes are present simultaneously in superposition.

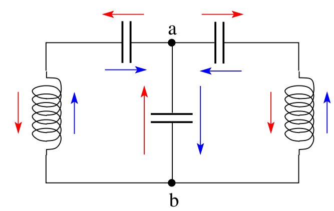
mode #1: $L\frac{dI}{dt} + \frac{Q}{C} + \frac{Q}{C} + L\frac{dI}{dt} = 0, \quad I = \frac{dQ}{dt}$

$\Rightarrow \frac{dI}{dt} = -\frac{Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \frac{1}{\sqrt{LC}}$



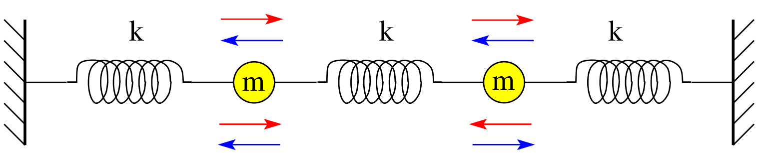
mode #2: $L\frac{dI}{dt} + \frac{Q}{C} + \frac{2Q}{C} = 0, \quad I = \frac{dQ}{dt}$

$\Rightarrow \frac{dI}{dt} = -\frac{3Q}{LC} \quad \Rightarrow \frac{d^2Q}{dt^2} = -\omega^2 Q, \quad \omega = \sqrt{\frac{3}{LC}}$



The mechanical analogue of this oscillator consists of two masses connected by three springs as shown. When this system oscillates in the slower mode, both masses move in the same direction all the time and the spring in the middle does not change its extension. The faster mode, by contrast, has the two masses move in opposite direction all the time, which now also compresses and stretches the spring in the middle.

The angular frequencies of the two modes are $\omega = \sqrt{\frac{k}{m}}, \quad \omega = \sqrt{\frac{3k}{m}}$.



RLC circuit:

Here we return to a single-mode electromagnetic oscillator and add a resistor in series to the capacitor and the inductor.

Loop rule: $RI + L\frac{dI}{dt} + \frac{Q}{C} = 0, \quad I = \frac{dQ}{dt}.$

Linear differential equation: $\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0.$

Initial conditions: $Q(0) = Q_{max}, \quad I(0) = \dot{Q}(0) = 0.$

The solution is a damped oscillation, for which there are three regimes:

– underdamped solution: $R^2 < \frac{4L}{C}, \quad \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

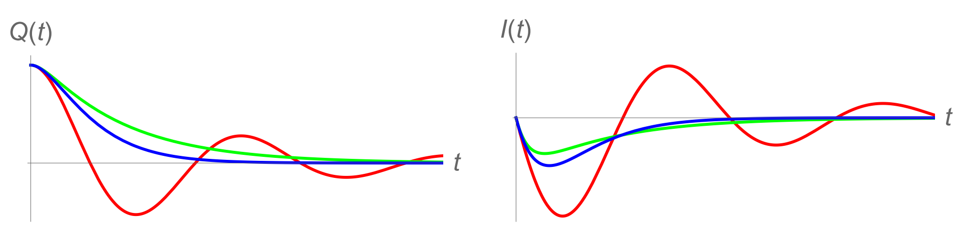
$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[\cos(\omega' t) + \frac{R}{2L\omega'}\sin(\omega' t)\right],$$

– overdamped motion: $R^2 > \frac{4L}{C}, \quad \Omega' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[\cosh(\Omega' t) + \frac{R}{2L\Omega'}\sinh(\Omega' t)\right],$$

– critically damped solution: $R^2 = \frac{4L}{C}, \quad \omega' \to 0 \text{ or } \Omega' \to 0$

$$Q(t) = Q_{max}\, e^{-Rt/2L}\left[1 + \frac{R}{2L}\,t\right].$$



Initially all energy is on the capacitor. In the underdamped regime, energy shuttles between capacitor and inductor as part of it is dissipated in the resistor. In the other two regimes, no energy returns to the capacitor.

The rate of energy dissipation in the resistor, RI^2, is proportional to the amount of energy storage in the inductor, $LI^2/2$.