Magnetostatics II [lln13]

Matter is dielectric. It responds to an external electric field **E** with a polarization **P**. Matter is also magnetic. It responds to an external magnetic field **B** with a magnetization **M**. The magnetic response of matter is more complex than the dielectric response.

Prominent types of magnetic materials in simplified characterizations:

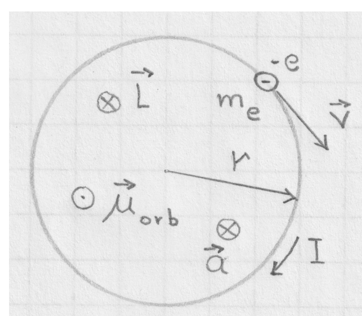
- ▷ *Diamagnetism*: The magnetization **M** is directed antiparallel to the magnetic field **B** present and vanishes when the field is turned off.
- ▷ *Paramagnetism*: The magnetization **M** is directed parallel to the magnetic field **B** present and vanishes when the field is turned off.
- ▷ *Ferromagnetism*: The magnetization **M** may have any direction relative to the magnetic field **B** and persists in the absence of a field.

A quantitative description of the magnetism of magnetic materials is inadequate in many respects without the use of quantum concepts. . [lln22][lln23]

Magnetic dipole moments of elementary particles:

The primary sources of magnetism in matter are the magnetic dipole moments associated with spin and atomic orbital angular momenta of electrons.

Orbital atomic magnetic moment of electron (semiclassically):



- Electron mass: $m_e \simeq 9.11 \times 10^{-31}$kg.
- Electron charge: $q = -e$.
- Radius of circular orbit: r.
- Electron speed: v.
- Orbital angular momentum: $L = m_e v r$.
- Loop area: $a = \pi r^2$.
- Effective current around loop : $I = -\dfrac{ev}{2\pi r}$.
- Orbital magnetic moment: $\boldsymbol{\mu}_{\text{orb}} = I\mathbf{a} = -\dfrac{e\mathbf{L}}{2m_e}$.
- Quantization of orbital angular momentum:

$$L = \sqrt{l(l+1)}\hbar, \; l = 0, 1, 2, \ldots, \quad L_z = m\hbar, \; m = -l, -l+1, \ldots, l.$$

- Orbital magnetic moment of electrons (z-component): $\mu_{\text{orb}} = -m\mu_B$.
- Bohr magneton: $\mu_B = \dfrac{e\hbar}{2m_e} \simeq 9.27 \times 10^{-24}\text{Am}^2$.

Spin magnetic moment of electron (z-component):

- Electron spin: $S_z = \pm\frac{1}{2}\hbar$.
- Gyromagnetic ratio: $\gamma \doteq \dfrac{g_\mathrm{e}\mu_\mathrm{B}}{\hbar} = \dfrac{g_\mathrm{e}e}{2m_\mathrm{e}}$.
- Spin g-factor: $g_\mathrm{e} \simeq 2$.
- Quantized spin magnetic moment: $\mu_\mathrm{spin} = -g_\mathrm{e}m_s\mu_\mathrm{B}, \quad m_s = \pm\frac{1}{2}$.

Atomic nuclei have much smaller magnetic moments, originating from the quantized spin magnetic moments of protons and neutrons.

- Proton mass: $m_\mathrm{p} \simeq 1.673 \times 10^{-27}\mathrm{kg}$.
- Neutron mass: $m_\mathrm{n} \simeq 1.675 \times 10^{-27}\mathrm{kg}$.
- Nuclear magneton: $\mu_\mathrm{N} = \dfrac{e\hbar}{2m_p} \simeq 5.05 \times 10^{-27}\mathrm{J/T} \simeq 5.44 \times 10^{-4}\mu_\mathrm{B}$.
- Spin of proton or neutron: $S_z = \pm\frac{1}{2}\hbar$.
- Gyromagnetic ratio: $\gamma \doteq \dfrac{g\mu_\mathrm{N}}{\hbar} = \dfrac{ge}{2m_\mathrm{p}}$.
- Proton g-factor: $g_\mathrm{p} \simeq 5.59$.
- Neutron g-factor: $g_\mathrm{n} \simeq -3.83$.
- Proton spin magnetic moment: $\mu_\mathrm{p} = \gamma\hbar|m_s| = g_\mathrm{p}\mu_\mathrm{N}|m_s| \simeq 2.79\,\mu_\mathrm{N}$.
- Neutron spin magnetic moment: $\mu_\mathrm{n} = \gamma\hbar|m_s| = g_\mathrm{n}\mu_\mathrm{N}|m_s| \simeq -1.91\,\mu_\mathrm{N}$.

Nuclear magnetic moments are important experimental probes in nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI) for the purpose of mapping magnetic-field environments created by electrons.

For comparison, the classical gyromagnetic ratio of a rotating massive object with uniform mass density and uniform charge density is $\gamma = Q/2M$, where Q is the total charge and M the total mass.

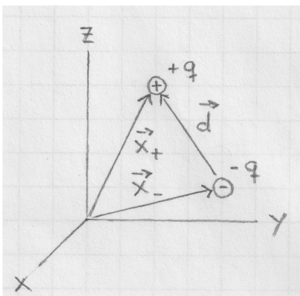
[lln22] The magnetism of atoms will be further investigated in a later module.

Electric and magnetic dipoles – commonalities and differences:

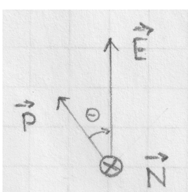
Region with charge density $\rho(\mathbf{x})$.

Electric dipole moment:
$\mathbf{p} = \int d^3x\, \mathbf{x}\, \rho(\mathbf{x})$.

Special case (pair of charges):
$\mathbf{p} = q(\mathbf{x}_+ - \mathbf{x}_-) = q\,\mathbf{d}$.



Torque:
$\mathbf{N} = \mathbf{p} \times \mathbf{E}, \quad N = pE \sin\theta$.



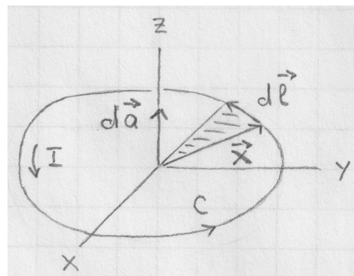
Potential energy:
$U = -\mathbf{p} \cdot \mathbf{E} = -pE \cos\theta$.

Force on $\mathbf{p}$ in field $\mathbf{E}(\mathbf{x})$:
$\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x}) = \nabla[\mathbf{p} \cdot \mathbf{E}(\mathbf{x})]$.

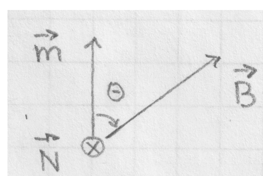
Region with current density $\mathbf{J}(\mathbf{x})$.

Magnetic dipole moment:
$\mathbf{m} = \frac{1}{2} \int d^3x\, \mathbf{x} \times \mathbf{J}(\mathbf{x})$.

Special case (current loop):
$\mathbf{m} = \frac{1}{2} I \oint_C \mathbf{x} \times d\mathbf{l} = I \int_S d\mathbf{a} = I\mathbf{a}$.



Torque:
$\mathbf{N} = \mathbf{m} \times \mathbf{B}, \quad N = mB \sin\theta$.



Potential energy:
$U = -\mathbf{m} \cdot \mathbf{B} = -mB \cos\theta$.

Force on $\mathbf{m}$ in field $\mathbf{B}(\mathbf{x})$:
$\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x}) = \nabla[\mathbf{m} \cdot \mathbf{B}(\mathbf{x})]$.

In a uniform field, the dipole experiences only a torque, which is the cause for reorientation (alignment with field). The dynamic response to torque is very different for electric and magnetic dipoles. [lex122]

In an inhomogeneous field, the dipole experiences a force in addition to a torque, which cause translocation and reorientation. The direction and magnitude of the force depend on the relative orientation between the dipole and the local field. [lex195][lex196]

Magnetization and bound currents:

The magnetization $\mathbf{M}(\mathbf{x})$ is a density of microscopic magnetic moments averaged over a mesoscopic length scale, assumed to be a differentiable function.

Magnetic dipole moment on mesoscopic scale: $d\mathbf{m} = \mathbf{M}(\mathbf{x}')d^3x'$.

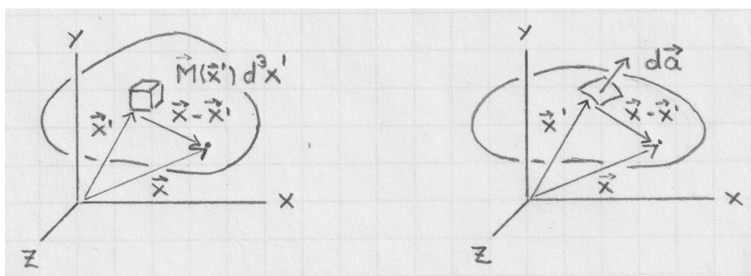
[lln12] Vector potential of magnetic dipole $d\mathbf{m}$: $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{d\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$.

Vector potential of magnetized macroscopic object:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\sum_i \frac{d\mathbf{m}_i(\mathbf{x}'_i)\times(\mathbf{x}-\mathbf{x}'_i)}{|\mathbf{x}-\mathbf{x}'_i|^3} = \frac{\mu_0}{4\pi}\int_V d^3x' \frac{\mathbf{M}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

Transformation of expression for $\mathbf{A}(\mathbf{x})$: [lex191]

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\int_V d^3x' \frac{\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \oint_S \frac{\mathbf{M}(\mathbf{x}')\times d\mathbf{a}'}{|\mathbf{x}-\mathbf{x}'|}\right].$$



[lex67][lex68]

Bound current densities associated with interior and surface magnetizations:

$$\mathbf{J}_\mathrm{b}(\mathbf{x}) \doteq \nabla\times\mathbf{M}(\mathbf{x}) \quad [\mathrm{A/m^2}] \quad \text{(interior)},$$
$$\mathbf{K}_\mathrm{b}(\mathbf{x}) \doteq \mathbf{M}(\mathbf{x})\times\hat{\mathbf{n}}, \quad \hat{\mathbf{n}} = \frac{d\mathbf{a}}{da} \quad [\mathrm{A/m}] \quad \text{(surface)}.$$

Vector potential generated by bound currents:

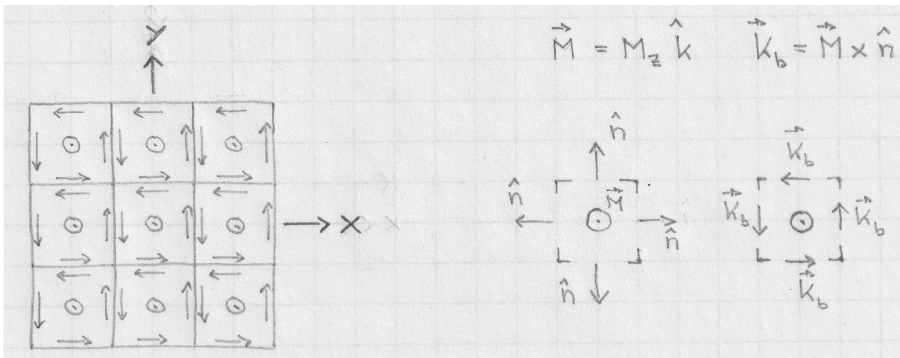
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\int_V d^3x' \frac{\mathbf{J}_\mathrm{b}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \oint_S d^2x' \frac{\mathbf{K}_\mathrm{b}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right].$$

Vector potential generated by conduction currents (for comparison):

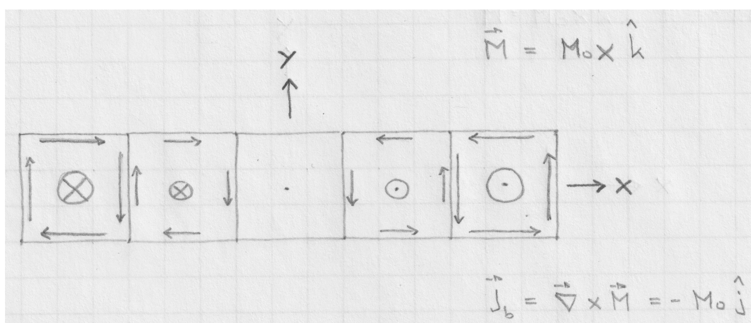
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left[\int_V d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \oint_S d^2x' \frac{\mathbf{K}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right].$$

A uniformly magnetized object is free of interior bound current density, $\mathbf{J}_\mathrm{b}(\mathbf{x}) = 0$. A nonzero surface current density, $\mathbf{K}_\mathrm{b}(\mathbf{x})$, is restricted to surfaces whose normal $\hat{\mathbf{n}}$ is not parallel to $\mathbf{M}(\mathbf{x})$.

If the magnetization is of the form $\mathbf{M} = M_z(z)\,\hat{\mathbf{k}}$, the interior current density still vanishes identically: $\mathbf{J}_\mathrm{b}(\mathbf{x}) \equiv 0$.



If the magnetization is of the form $\mathbf{M} = M_z(x, y)\,\hat{\mathbf{k}}$, the interior current density is of the form $\mathbf{J}_\mathrm{b} = (dM_z/dy)\,\hat{\mathbf{i}} - (dM_z/dx)\,\hat{\mathbf{j}}$.



Magnetic field H and magnetic induction B:

Ampère's law in the presence of free (conduction) current density $\mathbf{J}_\mathrm{f}(\mathbf{x})$ and bound current density $\mathbf{J}_\mathrm{b}(\mathbf{x}) = \nabla \times \mathbf{M}(\mathbf{x})$ due to a magnetized material:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{b}).$$

Distinguish between (fundamental) B-field, named *magnetic induction* or *magnetic flux density* or *magnetic field*, and (auxiliary) H-field, named *magnetic field*.

SI units: $\mathbf{B}$ [T], $\mathbf{H}$ [A/m].

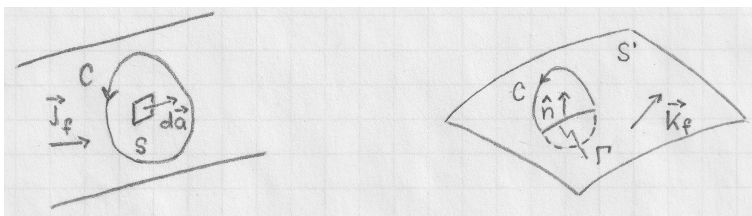
General relation: $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad \Rightarrow \; \nabla \times \mathbf{B} = \mu_0 \big[\underbrace{\nabla \times \mathbf{H}}_{\mathbf{J}_\mathrm{f}} + \underbrace{\nabla \times \mathbf{M}}_{\mathbf{J}_\mathrm{b}}\big]$.

Ampère's law for H-field:

- differential version: $\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f}$,
- integral version if a bulk current density $\mathbf{J}_\mathrm{f}$ is present:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_\mathrm{f} \cdot d\mathbf{a} = I_\mathrm{f}^{(en)}.$$

  The loop C encloses the open surface S.



- integral version if only a surface current density $\mathbf{K}_\mathrm{f}$ is present:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_\Gamma (\mathbf{K}_\mathrm{f} \times \hat{\mathbf{n}}) \cdot d\mathbf{l}' = I_\mathrm{f}^{(en)}.$$

  The path Γ with segments $d\mathbf{l}'$ is along the surface S' between the points where the loop C intersects the surface. The unit vector $\hat{\mathbf{n}}$ is normal to the suface S'.

[lex69]

Constitutive relations: $\mathbf{B} = \mu\mathbf{H}$ or $\mathbf{M} = \chi_\mathrm{m}\mathbf{H}$,

- linearity limited to weak fields,
- $\chi_\mathrm{m} > 0$: paramagnetic susceptibility,
- $\chi_\mathrm{m} < 0$: diamagnetic susceptibility,
- $\mu = \mu_0(1 + \chi_\mathrm{m})$: permeability of a material,
- $\kappa_\mathrm{m} \doteq \dfrac{\mu}{\mu_0} = 1 + \chi_\mathrm{m}$: relative permeability (used in ferromagnets),
- [lam16] tabulated data in additional materials.

Scalar magnetic potential:

Special circumstances in magnetism afford analogy with electrostatics:

▷ Electrostatics (general situations):

- Electrostatic field is irrotational.
- For linear dielectrics and away from free charges, the electric field is source-free.
- The (scalar) electric potential satisfies the Laplace equation.

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \ \mathbf{E} = -\nabla\Phi, \qquad \nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \ \nabla^2\Phi = 0.$$

▷ Magnetism (restricted situations):

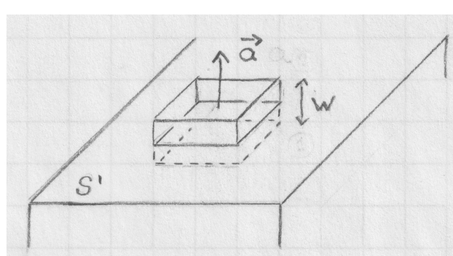
- In the absence of free currents, the H-field is irrotational.
- In a linear and homogeneous magnetic medium, the H-field is divergence-free.
- The magnetic field $\mathbf{H}$ is obtained from the gradient of a scalar magnetic potential Φ_m which satisfies the Laplace equation.

$$\nabla \times \mathbf{H} = 0 \quad \Rightarrow \ \mathbf{H} = -\nabla\Phi_\mathrm{m}, \qquad \nabla \cdot \mathbf{H} = 0 \quad \Rightarrow \ \nabla^2\Phi_\mathrm{m} = 0.$$

Boundary conditions:

▷ Magnetic monopoles are not known to exist, implying that the B-field is divergence-free: $\nabla \cdot \mathbf{B} = 0$. In consequence, the normal part $\mathbf{B}_\perp$ is continuous across any surface or interface S': [lex66][lex70] [lex201]

$$\lim_{w\to 0} \oint_S \mathbf{B} \cdot d\mathbf{a} = \Delta\mathbf{B}_\perp \cdot \mathbf{a} = 0 \quad \Rightarrow \ \Delta\mathbf{B}_\perp = 0.$$

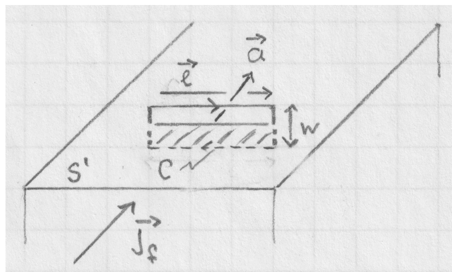


The same conclusion holds for the normal part of $\mathbf{H}$: $\Delta\mathbf{H}_\perp = 0$.

▷ The curl of the H-field is equal to the density of free current: $\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f}$. In consequence, the tangential part $\mathbf{H}_\parallel$ is continuous across any surface or interface S':

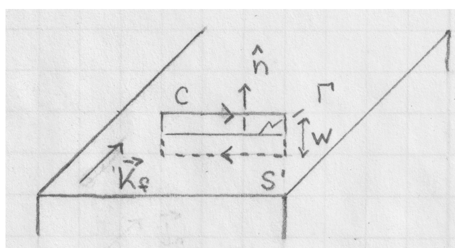
$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_\mathrm{f} \cdot d\mathbf{a}, \quad \lim_{w \to 0} \int_S \mathbf{J}_\mathrm{f} \cdot d\mathbf{a} = 0 \quad \Rightarrow \quad \lim_{w \to 0} \oint_C \mathbf{H} \cdot d\mathbf{l} = \Delta \mathbf{H}_\parallel \cdot \mathbf{l} = 0$$

$$\Rightarrow \quad \Delta \mathbf{H}_\parallel = 0.$$



▷ However, across a surface that carries a surface current of density $\mathbf{K}_\mathrm{f}$, the discontinuity of the tangential part $\mathbf{H}_\parallel$ can be nonzero:

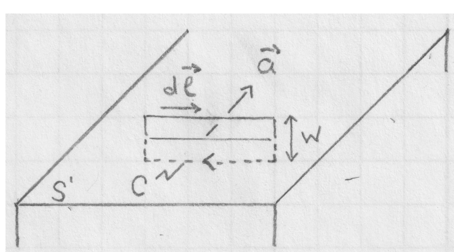
$$\lim_{w \to 0} \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_\Gamma \Delta \mathbf{H}_\parallel \cdot d\mathbf{l} = \int_\Gamma (\mathbf{K}_\mathrm{f} \times \hat{\mathbf{n}}) \cdot d\mathbf{l} \quad \Rightarrow \quad \Delta \mathbf{H}_\parallel = \mathbf{K}_\mathrm{f} \times \hat{\mathbf{n}}.$$



▷ Given that the (static) magnetic induction is the curl of the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$, we can apply Stokes' theorem to a flat loop straddling a surface or interface S' to infer that $\Delta \mathbf{A}_\parallel = 0$.

$$\int_S \mathbf{B} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{l}, \quad \lim_{w \to 0} \int_S \mathbf{B} \cdot d\mathbf{a} = 0 \quad \Rightarrow \quad \lim_{w \to 0} \oint_C \mathbf{A}_\parallel \cdot d\mathbf{l} = 0$$

$$\Rightarrow \quad \Delta \mathbf{A}_\parallel = 0.$$



▷ In the absence of free currents, $\mathbf{J}_\mathrm{f} = 0$, the magnetic field is irrotational, $\nabla \times \mathbf{H} = 0$. In consequence, the scalar magnetic potential Φ_m must be continuous across any surface or interface:

$$\Delta\Phi_\mathrm{m} = 0.$$

[lex72][lex73]

This last item is a constraint, not a boundary condition. However, it belongs into the same toolbox.

▷ Given the relations,

$$\nabla \times \mathbf{M} = \mathbf{J}_\mathrm{b}, \quad \nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f},$$

established previously and under the restriction pertaining to linear magnetic materials, we infer that bound currents and free currents necessarily make a joint appearance:

$$\mathbf{J}_\mathrm{b} = \chi_\mathrm{m}\mathbf{J}_\mathrm{f}.$$

Diamagnetism:

Diamagnetic response, like dielectric response, is universal in atomic matter. Diamagnetic materials are those that do not exhibit a different (typically stronger) magnetic response. Diamagnetism is characterized by a negative magnetic susceptibility.

Classical theories of diamagnetism are on very shaky ground. Langevin's classical model of diamagnetism works for individual atoms if we disregard the fact that the atomic structure is intrinsically quantum mechanical. The
[lln14] modeling requires concepts (Faraday's law) to be introduced later.

A proper statistical mechanical treatment of the classical diamagnetism made plausible for individual atoms wipes out the effect completely.

The (quantum mechanical) Van Vleck of theory diamagnetism will be pre-
[lln22] sented in a later module. A manifestation of strong diamagnetism is realized in superconductors, which again requires a quantum mechanical description.

Paramagnetism:

The permanent magnetic dipole moments $\mathbf{m}$ of unpaired electrons in atomic orbitals have random orientations, averaging the magnetization $\mathbf{M}$ to zero.

An external magnetic field $\mathbf{B}$ tends to align these microscopic magnetic moments $\mathbf{m}$ in field direction by a torque, $\mathbf{N} = \mathbf{m} \times \mathbf{B}$. The orientational potential energy, $U = -\mathbf{m} \cdot \mathbf{B}$, is lowest when the two vectors are parallel.

Thermal fluctuations are counteracting this alignment tendency, an effect captured in Curie's law. Statistical mechanical models of localized electron magnetic moments are known as Langevin paramagnetism and Brillouin
[lln22] paramagnetism.

The alignment of magnetic dipoles (e.g. electrons) in a magnetic field is akin
[lln5][lln12] to the alignment of electric dipoles (e.g. H_2O molecules) in an electric field.

Note again that microscopic magnetic or electric dipoles are intrinsically quantum mechanical objects.

The paramagnetism of (delocalized) band electrons is very different, known as Pauli paramagnetism. It has a much weaker temperature-dependence. The theory of Pauli paramagnetism requires quantum statistical modeling.

The process of orientational alignment with the external field is based on very different dynamics for electric and magnetic dipoles. [lex122]

Ferromagnetism:

The magnetic dipole moments of electrons in atoms are the source of paramagnetism and ferromagnetism. Their interaction is negligibly weak in paramagnets but very strong in ferromagnets.

Classical electricity and magnetism provide no source whatsoever for an interaction between microscopic magnetic dipoles of sufficient strength to explain the phenomenon of ferromagnetism as observed. The magnetic dipolar interaction is too weak by several orders of magnitude.

[lln23] The interaction which does the trick is known as *exchange interaction* between electron spins. It is non-magnetic in nature, caused by the interplay of the electrostatic repulsion between (negatively charged) electrons and the symmetry type of electronic wave functions (governed by the Pauli exclusion principle).

Permanent magnetism: phenomenon of persistent magnetization $\mathbf{M}$ not induced by magnetic fields $\mathbf{H}$, realized e.g. in bar magnets.

In ferromagnets, the relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ between magnetic induction, magnetic field, and magnetization is very complex.

Progression of complexity:

- Linear behavior as realized in diamagnets and paramagnets for weak magnetic fields: $B = \mu H$ with $\mu = \text{const}$.
- Nonlinear behavior as realized in paramagnets at stronger magnetic fields: $B = \mu(H)H$, incorporating saturation effects for magnetization. Quasistatic magnetization/demagnetization processes are reversible.
- Hysteretic behavior as realized in ferromagnets makes the functional dependence, $B = \mu(H)H$, dependent on initial state and on whether H is increasing or decreasing. Hysteresis loops are intrinsically irreversible, thus involving energy dissipation.

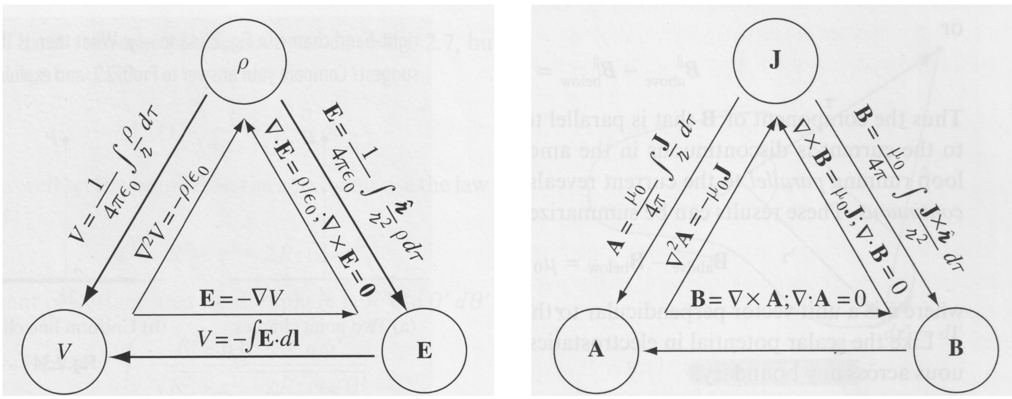
Ferromagnetic macrostates involve domains of uniformly magnetized regions. The formation of domains with magnetization in different directions is energetically favorable.

The magnetization process in ferromagnets involves the shift of domain boundaries. Domains with $\mathbf{M}$ in the direction of $\mathbf{H}$ grow at the expense of others.

A detailed description of ferromagnetic macrostates is a topic of solid-state physics and requires statistical mechanical modeling.

Sources – potentials – fields:

The mutual relationship between sources, potentials, and fields in electrostatics and magnetostatics has been summarized by Griffiths in two visually most pleasing diagrams.[1]



[images from Griffiths 2024]

The aesthetic appeal of commonalities and differences is quite enticing.

The triangular structures emphasize the direct links between each pair of three corner stones: source, potential, and field. All links except one can be formulated concisely in both directions.

Physically, the sources are ρ and $\mathbf{J}$, the potentials are Φ and $\mathbf{A}$, and the fields are $\mathbf{E}$ and $\mathbf{B}$. Mathematically, we have two scalar fields and four vector fields.

Not included are the auxiliary fields $\mathbf{D}$ and $\mathbf{H}$, introduced in the context of the dielectric and magnetic response of matter. Not represented either are the quantities with which we began: electric and magnetic forces.

As we move to electrodynamics in the next module, the two diagrams become interlinked into a single diagram, governed by the four Maxwell equations and
[lln14][lln15] the continuity equation, interrelating the six (time-dependent) fields.

In a relativistic formulation of electrodynamics, the two sources $\rho, \mathbf{J}$ are combined into a current 4-vector, the two potentials $\Phi, \mathbf{A}$ into a 4-vector poten-
[lln25] tial, and the fields $\mathbf{E}, \mathbf{B}$ into an electromagnetic field tensor.

The identities (in the diagrams above) of charge density and current density, of electric and magnetic fields, softens for observers in relative motion.

[1]Differences in notation include scalar potential, $V \to \Phi$, distance, $r \to |\mathbf{x} - \mathbf{x}'|$, and volume element, $d\tau \to d^3x'$.