Magnetostatics I [lln12]

Here we begin a discussion of situations involving static, i.e. time-independent magnetic fields. One source of such fields are steady electric currents, i.e. moving electric charges.

The term *magnetostatics* does not exactly have the same standing as the term *electrostatics* used previously.

Conduction currents are made of moving charged particles. Steady electric currents are consistent with time-independent charge densities on a mesoscopic length scale if hey are divergence-free.

It is common practice to discuss some effects of a static magnetic field before the sources of a static magnetic field are introduced.

Lorentz force:

In a region of space where a static electric field $\mathbf{E}$ and a static magnetic field $\mathbf{B}$ are present, a charged particle experiences the force,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Electrostatic limit: $\mathbf{v} = 0, \quad \mathbf{B} = 0 \quad \Rightarrow \mathbf{F} = q\mathbf{E}$.

Magnetic force on a charged particle:

Magnetic field: $\mathbf{B} = B\,\hat{\mathbf{k}}$ [T=N/Am] (Tesla).

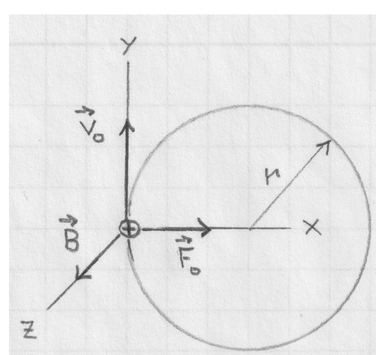
Initial velocity: $\mathbf{v}_0 = v_0\,\hat{\mathbf{j}}$ [m/s].

Initial force: $\mathbf{F}_0 = qv_0B\,\hat{\mathbf{i}}$ [N].

The orthogonality, $\mathbf{F} \perp \mathbf{v}$, conserves the speed v_0 .

Centripetal force required and provided: $\dfrac{mv_0^2}{r} = qv_0B$. [lex50][lex62] [lex71]

Radius of circular orbit: $r = \dfrac{mv_0}{qB}$.



Cyclotron frequency: $\omega = \dfrac{v_0}{r} = \dfrac{qB}{m}$.

Note that ω is independent of v_0 and r.

Magnetic force on current-carrying conductor:

A conductor of length L and cylindrical shape (e.g. a metal wire or a pipe filled with electrolyte) is positioned along the x-axis. A uniform magnetic field in y-direction is present. An applied voltage generates a uniform electric field inside the conductor, which drives a steady current.

Electric field: $\mathbf{E} = E\,\hat{\mathbf{i}}$ [N/C].

Charge carriers: q_i [C].

Drift velocities: $\mathbf{v}_i = v_{ix}\,\hat{\mathbf{i}}$ [m/s].

Number densities: n_i [m^{-3}].

Magnetic field: $\mathbf{B} = B\,\hat{\mathbf{j}}$ [T].

Magnetic force on charge carrier: $\mathbf{F}_i = q_i\mathbf{v}_i \times \mathbf{B} = q_i v_{ix} B\,\hat{\mathbf{k}}$ [N].

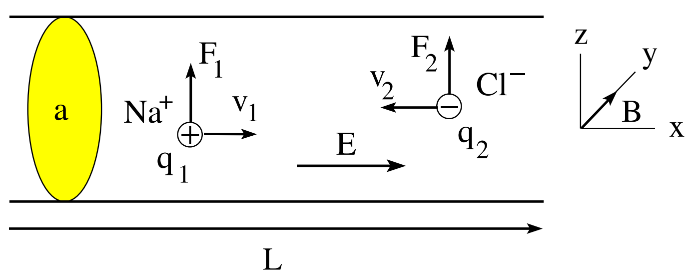
Current density: $\mathbf{J} = \sum_i q_i n_i \mathbf{v}_i = \sum_i q_i n_i v_{ix}\,\hat{\mathbf{i}} = J\,\hat{\mathbf{i}}$ [A/m^2].

Cross-sectional area: $\mathbf{a} = a\,\hat{\mathbf{i}}$ [m^2].

Current: $I = \int d\mathbf{a}\cdot\mathbf{J} = Ja = a\sum_i q_i n_i v_{ix}$ [A].

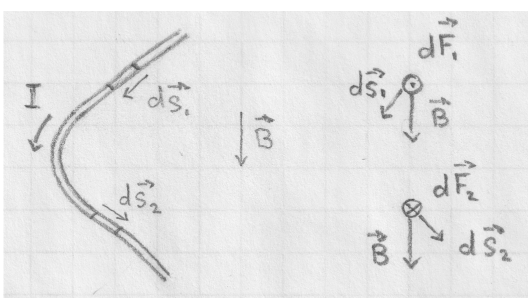
Length vector: $\mathbf{L} = L\,\hat{\mathbf{i}}$ (consistent with vector $\mathbf{a}$).

Force on conductor: $\mathbf{F} = aL\sum_i n_i \mathbf{F}_i = ILB\,\hat{\mathbf{k}} = I\mathbf{L}\times\mathbf{B}$ [N].



Generalization to a wire of more general shape:

$$\mathbf{F} = I\int_{\text{wire}} d\mathbf{s}\times\mathbf{B}.$$



Biot-Savart law:

Among the various sources of magnetic field, we consider here one of the most common. A wire carrying a steady current generates a static magnetic field in the space around it.

The Biot-Savart law is an application of more fundamental laws to this scenario. It is of practical value for the calculation of the magnetic field $\mathbf{B}$ generated by a steady current I through a thin wire of arbitray shape:

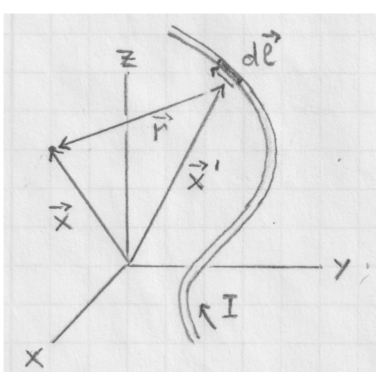
$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{I d\mathbf{l}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{I d\mathbf{l}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{1}$$

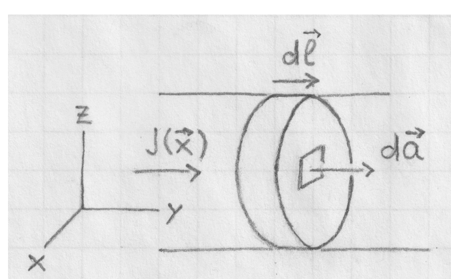
- ▷ $\mu_0 = 4\pi\times 10^{-7}\text{Tm/A}$: permeability constant,
- ▷ $\mathbf{x}'$: position of source point,
- ▷ $\mathbf{x}$: position of field point,
- ▷ $d\mathbf{l}$: infinitesimal wire segment in current direction,
- ▷ $\mathbf{r} = \mathbf{x}-\mathbf{x}'$: distance vector pointing from source point to field point,
- ▷ $\hat{\mathbf{r}} = \mathbf{r}/r$: unit vector pointing from source point to field point.

Generalization of the Biot-Savart law to conductors of arbitrary shape with given current density $\mathbf{J}(\mathbf{x}')$ inside:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{\text{cond.}} d^3x'\,\frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{2}$$

Bridge from (2) to (1):[1] $d^3x\,\mathbf{J} \rightsquigarrow \underbrace{d\mathbf{a}\cdot d\mathbf{l}}_{d^3x}\,\mathbf{J} \rightsquigarrow \underbrace{d\mathbf{a}\cdot\mathbf{J}}_{I}\,d\mathbf{l}$. [lex51][lex52]
[lex59][lex60]
[lex129]





[1]This transformation is not generally valid. It relies on the assumption that $d\mathbf{a}, \mathbf{l}, \mathbf{J}$ are vectors of the same direction.

Magnetic flux:

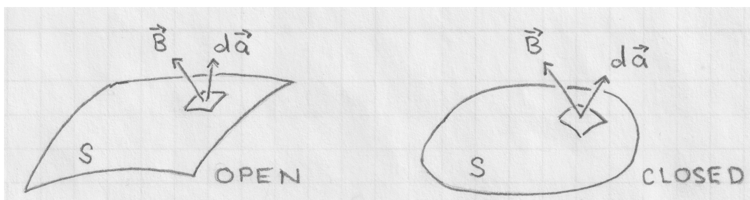
Consider a surface S of arbitrary shape divided into infinitesimal elements of area da, represented as vectors $d\mathbf{a}$ directed perpendicular to the surface.

For open surfaces one of two directions is chosen. For closed surfaces the convention is that $d\mathbf{a}$ points toward the outside.

Magnetic flux through an arbitrary surface S:

$$\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{a} \quad [\mathrm{Tm}^2] = [\mathrm{Wb}] \quad (\mathrm{Weber}).$$

The variation in time of magnetic flux through open surfaces plays an important role in magnetic induction (a later topic).



Gauss's law for the magnetic field:

Integral version: $\oint_S \mathbf{B} \cdot d\mathbf{a} = 0.$

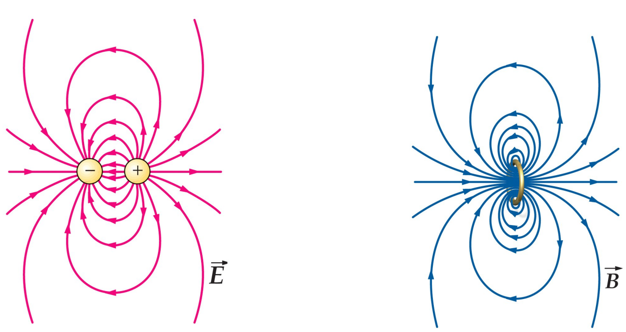
In words: the magnetic flux through any closed surface vanishes identically.

Implication: there are no magnetic charges (monopoles).

Differential version: $\nabla \cdot \mathbf{B} = 0.$

The two versions are related by Gauss's theorem.

Electrostatic field lines begin and end at electric charges or at infinity. Magnetic field lines have no ends. The ones shown close in themselves. There are electrodynamic field lines that also close in themselves (a later topic).



Ampère's law restricted to steady states:

Assumptions: static magnetic fields, steady electric currents.

Differential version: $\nabla \times \mathbf{B} = \mu_0 \, \mathbf{J}$.

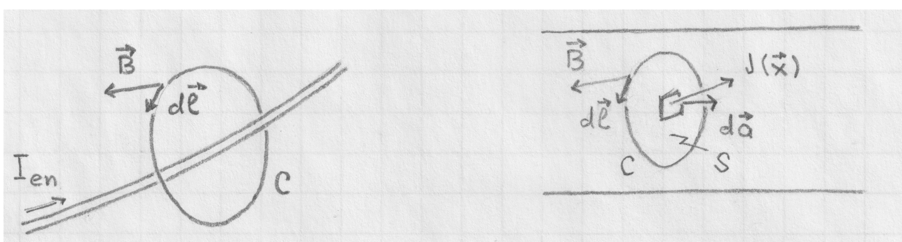
Integral version: $\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\rm en}$.

The open surface S is surrounded by the loop C.

The two versions of Ampère's law are related by Stokes' theorem.

The current $I_{\rm en}$, enclosed by loop C, is the flux of the current density $\mathbf{J}$ through the open surface S.

Right-hand rule: If the loop C is traversed cw (ccw), then the area vector is directed in (out). [lex65]



Magnetostatics and electrostatics:

(B1) $\nabla \cdot \mathbf{B} = 0$: the magnetic field is solenoidal (has zero diveergence);

(E1) $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$: electric charges control the divergence of $\mathbf{E}$;

(B2) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$: conduction currents control the curl of $\mathbf{B}$;

(E2) $\nabla \times \mathbf{E} = 0$: the electric field is irrotational (has zero curl).

Equations (B1) and (E1) hold in all situations. Equations (B2) and (E2) hold only for time-independent fields.

Conservation of electric charge implies that any charge entering a region V causes a flux of current density through its surface S:

$$\frac{dQ}{dt} = \int_V d^3x \, \frac{\partial \rho}{\partial t} = -\oint_S d\mathbf{a} \cdot \mathbf{J} = -\int_V d^3x \, \nabla \cdot \mathbf{J} \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Charge density ρ and current density $\mathbf{J}$ must satisfy the continuity equation.

In magnetostatics we have $\dfrac{\partial \rho}{\partial t} = 0$, which implies $\nabla \cdot \mathbf{J} = 0$.

Consistency of the Biot-Savart law with Ampère's law:

Steady current I around loop C'.

Magnetic field: $\mathbf{B}(\mathbf{x}) = \dfrac{\mu_0 I}{4\pi} \oint_{C'} \dfrac{d\mathbf{x}' \times \hat{\mathbf{r}}}{r^2}$ (Biot-Savart law).

Integral around Amperian loop C: $\gamma \doteq \int_C d\mathbf{x} \cdot \mathbf{B}(\mathbf{x})$.

Distance vector: $\mathbf{r} \doteq \mathbf{x} - \mathbf{x}'$ (from source point to field point).

Unit vector: $\hat{\mathbf{r}} \doteq \dfrac{\mathbf{r}}{r}$.

Mathematical identity: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$.

$$\Rightarrow \ \gamma = \frac{\mu_0 I}{4\pi} \oint_C \oint_{C'} \frac{(d\mathbf{x}' \times \hat{\mathbf{r}}) \cdot d\mathbf{x}}{r^2} = \frac{\mu_0 I}{4\pi} \oint_C \oint_{C'} \frac{(d\mathbf{x} \times d\mathbf{x}') \cdot \hat{\mathbf{r}}}{r^2}.$$

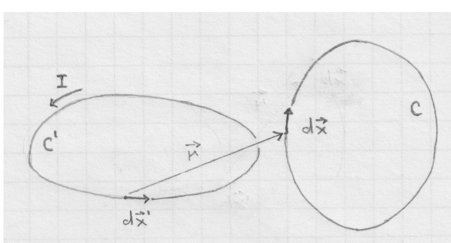
Vector $\mathbf{r}$ traces closed surface S.

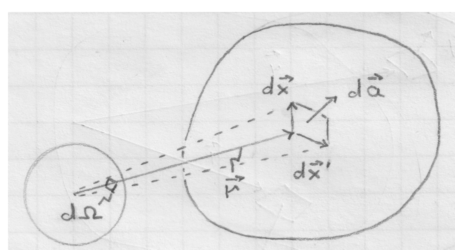
Element of area vector on S: $d\mathbf{a} = d\mathbf{x} \times d\mathbf{x}'$.

Element of solid angle on S: $d\Omega = \dfrac{d\mathbf{a} \cdot \hat{\mathbf{r}}}{r^2}$.

$$\Rightarrow \ \gamma = \frac{\mu_0 I}{4\pi} \oint_S \frac{d\mathbf{a} \cdot \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \oint_S d\Omega.$$

Case #1: Loops C and C' are not interlinked.





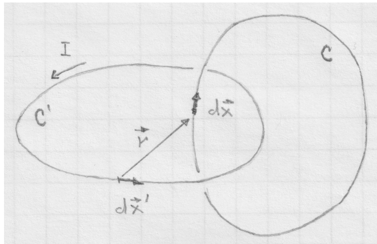
Vector $\mathbf{r}$ traces surface S from a point outside.

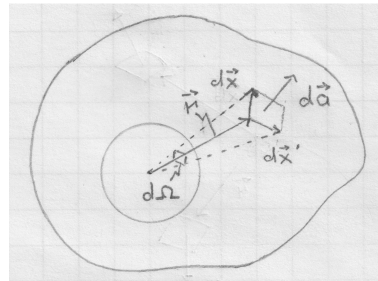
All elements of solid angle within a limited range contribute twice, with opposite sign.

Solid angle: $\oint_S d\Omega = 0 \quad \Rightarrow \ \gamma = 0$.

$\Rightarrow \ \int_C d\mathbf{x} \cdot \mathbf{B}(\mathbf{x}) = 0$ (Ampère's law if loop encloses no net current).

Case #2: Loops C and C' are interlinked.





Vector $\mathbf{r}$ traces surface S from a point inside.

All elements of solid angle over complete range contribute once, with positive sign.

Solid angle: $\oint_S d\Omega = 4\pi \quad \Rightarrow \; \gamma = \mu_0 I.$

$\Rightarrow \quad \int_C d\mathbf{x}\cdot\mathbf{B}(\mathbf{x}) = \mu_0 I \quad$ (Ampère's law if loop encloses current I).

The above demonstration shows that the Biot-Savart law is consistent with the restricted version of Ampère's law, known to hold if there are no time-dependent electric or magnetic fields involved.

Vector potential:

The unique specification of the vector potential $\mathbf{A}$ in the restricted context of magnetostatics has three parts:

- ▷ relation to magnetic field: $\nabla\times\mathbf{A} = \mathbf{B}$,
- ▷ Coulomb gauge condition:[2] $\nabla\cdot\mathbf{A} = 0$,
- ▷ boundary conditions: e.g. $\lim_{|\mathbf{x}|\to\infty}\mathbf{A}(\mathbf{x}) = 0$.

Consequence of Ampère's law: $\nabla\times(\nabla\times\mathbf{A}) = \mu_0\mathbf{J}$.

Mathematical identity applied to vector potential:

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\underbrace{(\nabla\cdot\mathbf{A})}_{0} - \nabla^2\mathbf{A} = -\nabla^2\mathbf{A}.$$

Poisson equation for vector potential: $-\nabla^2\mathbf{A}(\mathbf{x}) = \mu_0\mathbf{J}(\mathbf{x})$.

[2]The concept of gauge invariance will be discussed in the context of electrodynamics [lln15]. In the context of magnetostatics, the two major gauge conditions are equivalent.

Integral expression:[3] $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$.

The Poisson equation follows by use of $\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.

Biot-Savart result: $\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$.

The Biot-Savart integral expression for $\mathbf{B}(\mathbf{x})$ follows from the integral expression for the vector potential $\mathbf{A}(\mathbf{x})$ via $\mathbf{B} = \nabla\times\mathbf{A}$.

- Use identity: $\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$.
- Use identity: $\nabla\times[f\mathbf{J}(\mathbf{x}')] = f[\nabla\times\mathbf{J}(\mathbf{x}')] - \mathbf{J}(\mathbf{x}')\times(\nabla f), \quad f \doteq \frac{1}{|\mathbf{x}-\mathbf{x}'|}$.
- Curl acts on variable $\mathbf{x}$, implying $\nabla\times\mathbf{J}(\mathbf{x}') = 0$.
- Consequence: $\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) = -\mathbf{J}(\mathbf{x}')\times\left(\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)$.

$$\Rightarrow\ \mathbf{B}(\mathbf{x}) = \nabla\times\underbrace{\left[\frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]}_{\mathbf{A}(\mathbf{x})} = -\frac{\mu_0}{4\pi}\int d^3x' \mathbf{J}(\mathbf{x}')\times\underbrace{\left(\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)}_{\text{use identity}}$$
$$= \frac{\mu_0}{4\pi}\int d^3x' \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

Vector potential for uniform magnetic field: $\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B}_0\times\mathbf{x}$.

$$\nabla\times\mathbf{A} = \frac{1}{2}\nabla\times(\mathbf{B}_0\times\mathbf{x})$$
$$= \frac{1}{2}\big[\mathbf{B}_0\underbrace{(\nabla\cdot\mathbf{x})}_{3} - \mathbf{x}\underbrace{(\nabla\cdot\mathbf{B}_0)}_{0} + \underbrace{(\mathbf{x}\cdot\nabla)\mathbf{B}_0}_{0} - \underbrace{(\mathbf{B}_0\cdot\nabla)\mathbf{x}}_{\mathbf{B}_0}\big] = \mathbf{B}_0.$$

A region of uniform magnetic field is free of electric currents:

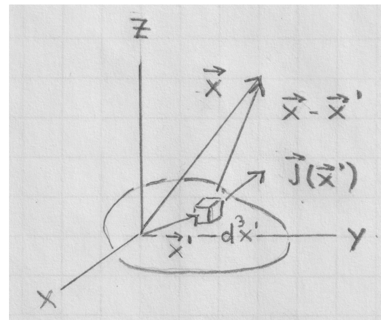
$$\mathbf{A}(\mathbf{x}) = \frac{1}{2}\mathbf{B}_0\times\mathbf{x} \quad\Rightarrow\ \nabla^2\mathbf{A} = 0 \quad\Rightarrow\ \mathbf{J} = 0.$$

[3]Note of caution: Only for Cartesian coordinates can this vector equation be split into equations for components, e.g. for $A_x(\mathbf{x})$ and $J_x(\mathbf{x}')$ etc. Integrals of vector quantities expressed in curvilinear coordinates require extreme care.

Magnetic dipole moment:

Consider a localized source of steady current specified by a current density $\mathbf{J}(\mathbf{x}')$ flowing in a region V of finite extension.

Vector potential: $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V d^3x' \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$.



Long-distance asymptotics associated with monopole, dipole, ...

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{x}'}{r^2} + \mathrm{O}\left(\frac{r'^2}{r^3}\right),$$

Leading term: $\mathbf{A}(\mathbf{x})_{\text{mon}} \doteq \frac{1}{r}\int_V d^3x' \mathbf{J}(\mathbf{x}')$,

- Mathematical identity: $\nabla\cdot(x_i\mathbf{J}) = x_i(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\nabla x_i$.
- Note change in notation: $x \to x_1, \quad y \to x_2, \quad z \to x_3$.
- Steady currents are non-divergent: $\nabla\cdot\mathbf{J} = 0$.
- Gradients: $\nabla x_1 = \hat{\mathbf{i}}, \quad \nabla x_2 = \hat{\mathbf{j}}, \quad \nabla x_3 = \hat{\mathbf{k}}$.
- Consequence: $\nabla\cdot(x_i\mathbf{J}) = J_i$.
- Use Gauss's theorem and that region V has finite extension:

$$\int_V d^3x' J_i(\mathbf{x}') = \int_V d^3x' \, \nabla'\cdot[x_i'\mathbf{J}(\mathbf{x}')] = \lim_{r\to\infty}\oint_S d\mathbf{a}\cdot[x_i'\mathbf{J}(\mathbf{x}')] = 0.$$

- Steady currents are no source of magnetic monopoles: $\mathbf{A}(\mathbf{x})_{\text{mon}} = 0$.

Next leading term: $\mathbf{A}(\mathbf{x})_{\text{dip}} \doteq \frac{\mu_0}{4\pi r^2}\int_V d^3x' \, \mathbf{J}(\mathbf{x}')\hat{\mathbf{r}}\cdot\mathbf{x}'$.

Magnetic dipole moment: $\mathbf{m} \doteq \frac{1}{2}\int_V d^3x' \, \mathbf{x}'\times\mathbf{J}(\mathbf{x}')$.

If $\hat{\mathbf{r}}$ is a (fixed) unit vector pointing to a distant field point $\mathbf{x}$, the following relation can be proven to hold: [lex190]

$$\int_V d^3x' \, \mathbf{J}(\mathbf{x}')\hat{\mathbf{r}}\cdot\mathbf{x}' = \mathbf{m}\times\hat{\mathbf{r}} = \frac{\mathbf{m}\times\mathbf{x}}{r}.$$

Vector potential of magnetic dipole: $\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$.

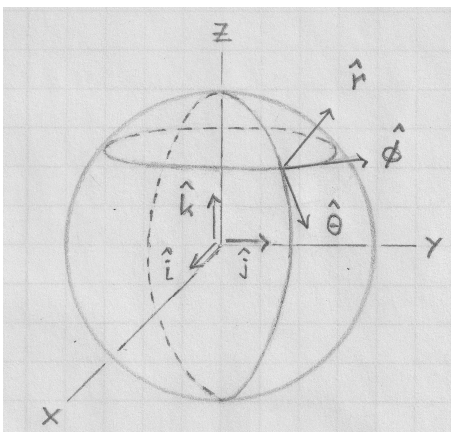
This is the leading term of the vector potential at a distant field point generated by a local current distribution.

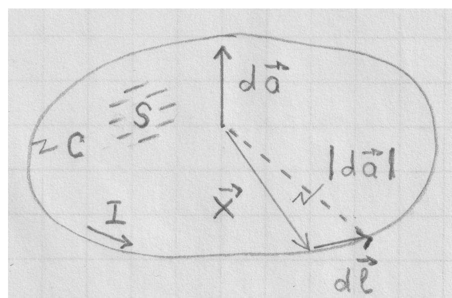
Magnetic dipole field: $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{[3\hat{\mathbf{r}}\,(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}]}{r^3}$.

Expressions in spherical coordinates:

[lex36]

- magnetic dipole: $\mathbf{m} = m\,\hat{\mathbf{k}} = m\cos\theta\,\hat{\mathbf{r}} - m\sin\theta\,\hat{\boldsymbol{\theta}}$,
- position: $\mathbf{x} = r\,\hat{\mathbf{r}}$,
- vector potential: $\mathbf{A}(\mathbf{x}) = \frac{\mu_0 m}{4\pi r^2}\,\sin\theta\,\hat{\boldsymbol{\phi}}$,
- dot products: $\mathbf{m}\cdot\hat{\mathbf{r}} = m\cos\theta, \quad \mathbf{m}\cdot\hat{\boldsymbol{\theta}} = -m\sin\theta$,
- magnetic field: $\mathbf{B}(\mathbf{x}) = \frac{\mu_0 m}{4\pi r^3}\big[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\big]$.





Magnetic dipole moment of plane current loop:

- Wire segment in current direction: $d\mathbf{l}$.
- Current I is flux of current density $\mathbf{J}$.
- Vector quantity equivalence for wire: $\mathbf{J}(\mathbf{x})d^3x = Id\mathbf{l}$.
- Triangular element of loop area: $d\mathbf{a} = \frac{1}{2}\mathbf{x}\times d\mathbf{l}$.
- Area vector of loop: $\mathbf{a}$.
- Magnetic dipole moment: [lex63][lex64] [lex78]

$$\mathbf{m} \doteq \frac{1}{2}\int d^3x\,\mathbf{x}\times\mathbf{J}(\mathbf{x}) = \frac{I}{2}\oint_C \mathbf{x}\times d\mathbf{l} = I\int_S d\mathbf{a} = I\mathbf{a}.$$

Torque and force acting on magnetic dipole:

Torque $\mathbf{N}$ on loop in uniform magnetic field: $\mathbf{B}$:

- Current segment: $I d\mathbf{x}$.
- Force on current segment: $d\mathbf{F} = I d\mathbf{x} \times \mathbf{B}$
- Torque on plane current loop: $\mathbf{N} = \oint_C \mathbf{x} \times d\mathbf{F} = \oint_C \mathbf{x} \times (I d\mathbf{x} \times \mathbf{B})$.
- Exact differential: $d[\mathbf{x} \times (\mathbf{x} \times \mathbf{B})] = \mathbf{x} \times (d\mathbf{x} \times \mathbf{B}) + d\mathbf{x} \times (\mathbf{x} \times \mathbf{B})$.
- Loop integral of exact differential vanishes.

$$\Rightarrow\; I \oint_C \mathbf{x} \times (d\mathbf{x} \times \mathbf{B}) = -I \oint_C d\mathbf{x} \times (\mathbf{x} \times \mathbf{B}).$$

$$\Rightarrow\; \mathbf{N} = \frac{I}{2} \oint_C \Big[\mathbf{x} \times (d\mathbf{x} \times \mathbf{B}) \underbrace{-d\mathbf{x} \times (\mathbf{x} \times \mathbf{B})}_{+d\mathbf{x} \times (\mathbf{B} \times \mathbf{x})}\Big].$$

- Mathematical identity: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

$$\Rightarrow\; \mathbf{N} = -\frac{I}{2} \oint_C \mathbf{B} \times (\mathbf{x} \times d\mathbf{x}) = \frac{I}{2} \oint_C (\mathbf{x} \times d\mathbf{x}) \times \mathbf{B} = \mathbf{m} \times \mathbf{B}.$$

Orientational potential energy U in uniform magnetic field $\mathbf{B}$:

- Directional change of magnetic moment: $d\mathbf{m} = d\boldsymbol{\theta} \times \mathbf{m}$.
- Increment in potential energy: $dU = -\mathbf{N} \cdot d\boldsymbol{\theta}$.

$$\Rightarrow\; dU = -(\mathbf{m} \times \mathbf{B}) \cdot d\boldsymbol{\theta} = -(d\boldsymbol{\theta} \times \mathbf{m}) \cdot \mathbf{B} = -d\mathbf{m} \cdot \mathbf{B}.$$

$$\Rightarrow\; U = -\mathbf{m} \cdot \mathbf{B}.$$

[lex130][lex131]
[lex133][lex136]
[lex148]

Force $\mathbf{F}(\mathbf{x})$ on magnetic dipole in nonuniform magnetic field $\mathbf{B}(\mathbf{x})$

$$\mathbf{F}(\mathbf{x}) = -\nabla U = \nabla[\mathbf{m} \cdot \mathbf{B}(\mathbf{x})].$$