Electric Currents I [lln11]

Our initial move away from *electrostatics* toward *electrodynamics* describes electric currents, which are electric charges in motion.

Microscopically, electric currents in metallic conductors or electrolytes are produced by drifting charged particles (electrons, protons, ions).

Electric current density:

The fundamental quantity describing electric currents is the current density:

$$\mathbf{J}(\mathbf{x}) = \sum_i q_i\, n_i(\mathbf{x})\mathbf{v}_i(\mathbf{x}) \quad [\mathrm{Cs^{-1}m^{-2}}] = [\mathrm{Am^{-2}}],$$

- ▷ q_i: charge of mobile charge carrier [C],
- ▷ $n_i(\mathbf{x})$: number density of charge carrier [$\mathrm{m^{-3}}$],
- ▷ $\mathbf{v}_i(\mathbf{x})$: drift velocity field of charge carrier [$\mathrm{ms^{-1}}$],
- ▷ $\rho(\mathbf{x}) = \sum_i q_i\, n_i(\mathbf{x})$: charge density [$\mathrm{Cm^{-3}}$].

Electric current:

The current is the flux quantity associated with current density:

$$I = \int_S \mathbf{J} \cdot d\mathbf{a} \quad [\mathrm{A}].$$

- ▷ S: open surface (e.g. across wire) or a closed surface.
- ▷ $d\mathbf{a}$: element of area on surface S.
- ▷ The direction of the area vector $d\mathbf{a}$ specifies the current direction through an open surface. Consequence: the current "direction" through a wire is always a matter of choice.
- ▷ By convention, any area vector $d\mathbf{a}$ on a closed surface points toward the outside.

Charge conservation:

Differential version: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$ (continuity equation).

Integral version: $\displaystyle\oint_S \mathbf{J} \cdot d\mathbf{A} = -\frac{d}{dt}\int_V d^3x\rho(\mathbf{x})$.

We employ Gauss's theorem to connect the two versions. The current flowing out of V through S reduces the net charge inside.

Ohm's law:

A device of arbitrary shape with two terminals is a resistor if the current I through the device and voltage V across it satisfy Ohm's law,

$$V = RI, \quad R = \text{const} \quad [\Omega] = [\mathrm{V/A}].$$

On the level of conducting materials, Ohm's law originates in a linear relation between (local) electric field and (local) current density:

$$\mathbf{J}(\mathbf{x}) = \sigma\mathbf{E}(\mathbf{x}) \quad \text{or} \quad \mathbf{E}(\mathbf{x}) = \rho\mathbf{J}(\mathbf{x}).$$

Resistivity:[1] $\rho = \dfrac{E}{J} \quad \left[\dfrac{\mathrm{V/m}}{\mathrm{A/m^2}}\right] = [\Omega\mathrm{m}]$.

Conductivity: $\sigma = \dfrac{1}{\rho} \quad \left[(\Omega\mathrm{m})^{-1}\right]$.

[lam16] Empirical data for material property ρ are compiled separately.

Resistance of a wire of length L and cross sectional area A, made of a conducting material with resistivity ρ:

- ▷ Current density: $J = \dfrac{E}{\rho} \quad [\mathrm{A/m^2}]$.
- ▷ Current: $I = JA \quad [\mathrm{A}]$.
- ▷ Voltage: $V = EL \quad [\mathrm{V}]$.
- ▷ Resistance: $R = \dfrac{V}{I} = \dfrac{\rho L}{A} \quad [\Omega]$.

[1]The symbol ρ, commonly used for *charge density*, is used here for *resistivity*. Likewise, the symbol σ, commonly used for *surface charge density*, is used here for *conductivity*.

Electric field driving steady current:

Cylindrical wire with uniform conductivity σ carries steady current I.

Use cylindrical coordinates: r, ϕ, z

Current is driven by electrostatic field $\mathbf{E}$ inside wire.

Electrostatic field is irrotational: $\nabla \times \mathbf{E} = 0$.

Current density has azimuthal and translational symmetry: $\mathbf{J} = J_z(r)\,\hat{\mathbf{z}}$.

$\mathbf{J} = \sigma\mathbf{E}$ and $\nabla \times \mathbf{E} = 0$ imply $\nabla \times \mathbf{J} = 0$.

Curl in cylindrical coordinates:

$$\nabla \times \mathbf{J} = \left(\frac{1}{r}\frac{\partial J_z}{\partial \phi} - \frac{\partial J_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial J_r}{\partial z} - \frac{\partial J_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial (rJ_\phi)}{\partial r} - \frac{\partial J_r}{\partial \phi}\right)\hat{\mathbf{z}}.$$

Irrotational $\mathbf{J}$ implies $\dfrac{dJ_z}{dr} = 0$.

Consequence: $\mathbf{J} = J_z\,\hat{\mathbf{z}}$ (uniform current density).

Consequence: $\mathbf{E} = E_z\,\hat{\mathbf{z}}$ (uniform electric field).

Source of electrostatic $\mathbf{E}$ can only be a charge distribution somewhere.

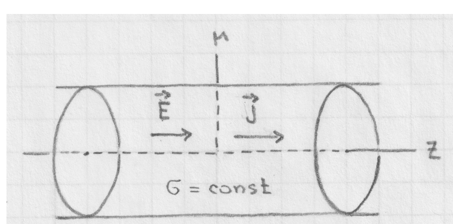
Steady state implies $\dfrac{\partial \rho}{\partial t} = 0$.

Continuity equation implies $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t} = 0$.

Relations $\mathbf{J} = \sigma\mathbf{E}$ and $\nabla \cdot \mathbf{J} = 0$ imply $\nabla \cdot \mathbf{E} = 0$.

Gauss's law implies zero charge density, $\rho = 0$, inside wire.

Consequence: source of $\mathbf{E}$ must be surface charge distribution along the wire.



Inhomogeneous resistivity permits nonzero volume charge density. [lex33]

Classical model of conductivity:

Single species of mobile charge carriers:

– number density: n $[\mathrm{m}^{-3}]$.
– charge: q [C].
– average velocity: $\langle \mathbf{v} \rangle$ [m/s].

Current density: $\mathbf{J} = qn\langle \mathbf{v} \rangle$ $[\mathrm{A/m}^2]$.

Average speed v_0 of charge carrier:

– classical Drude model: $\frac{1}{2}mv_0^2 = \frac{3}{2}k_\mathrm{B}T_\mathrm{rm}$ (room temp.: $T_\mathrm{rm} \sim 300\mathrm{K}$).

– metallic conductor: $\frac{1}{2}mv_0^2 = k_\mathrm{B}T_\mathrm{F}$ (Fermi temp.: $T_\mathrm{F} \gtrsim 30,000\mathrm{K}$).

Mean free path between collisions: λ [m].

Time between collisions: $\tau = \lambda/v_0$ [s].

Acceleration of charge carrier: $a = \dfrac{qE}{m}$ $[\mathrm{m/s}^2]$.

Drift velocity: $\mathbf{v}_\mathrm{d} = \langle \mathbf{v} \rangle$, $v_\mathrm{d} = a\tau = \dfrac{qE}{m}\dfrac{\lambda}{v_0}$ [m/s].

Conductivity: $\sigma = \dfrac{J}{E} = \dfrac{nqv_\mathrm{d}}{E} = \dfrac{nq^2\lambda}{mv_0}$ $\left[(\Omega\mathrm{m})^{-1}\right]$.

Temperature coefficient: $\alpha \doteq \dfrac{1}{\rho}\dfrac{d\rho}{dT}$ $[\mathrm{K}^{-1}]$ (typically $\sim 10^{-3}\mathrm{K}^{-1}$).

Power dissipation in resistive materials:

Work per unit time done by electric field on drifting charge carriers is electric power converted into heat.

$$dP = dQ\,\mathbf{v}_\mathrm{d} \cdot \mathbf{E} = qn\mathbf{v}_\mathrm{d} \cdot \mathbf{E}\,d^3x = \mathbf{J} \cdot \mathbf{E}\,d^3x = \rho J^2 d^3x \quad \Rightarrow \; P = \int d^3x\,\rho J^2.$$

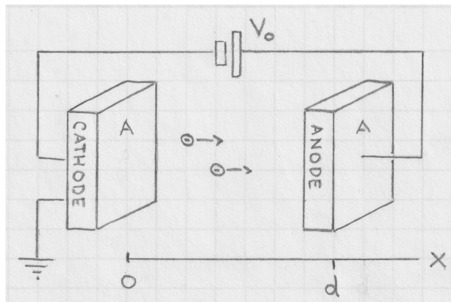
Application to resistor in the form of a wire of length l and cross-sectional area A with uniform resistivity ρ:

$$P = \rho\left(\frac{I}{A}\right)^2 Al = \left(\frac{\rho\,l}{A}\right) I^2 = RI^2.$$

Maintaining a steady current through a conductor requires a steady supply of power.

Electric current in vacuum-tube:

The cathode (at $x = 0$) is grounded and the anode (at $x = d$) is at electric potential V_0. Both have cross-sectional area A. Heating up the cathode facilitates the thermal emission of electrons and produces an electric current.



Electric potential without current is linear: $\Phi_0(x) = V_0\,\dfrac{x}{d}$.

A current modifies the potential to a function $\Phi(x)$ with a nonlinear profile.

Relevant quantities:

- electric potential is monotonically increasing: $\Phi(x)$ [V]
- charge density of electrons is negative: $\rho(x) = -en(x)$ [C/m^3],
- electron velocity is positive and increasing: $v(x)$ [m/s],
- current density is negative: $J(x) = \rho(x)v(x)$ [A/m^2].

Relations for the case of a steady current:

(1) Energy conservation: $\dfrac{1}{2}mv^2 = e\,\Phi \quad \Rightarrow \; v(x) = \sqrt{\dfrac{2e\,\Phi(x)}{m}}$.

(2) Continuity equation: $\dfrac{\partial \rho}{\partial t} = -\dfrac{dJ}{dx} = 0 \quad \Rightarrow \; J$ is independent of x.

(3) Poisson equation: $\dfrac{d^2\Phi}{dx^2} = -\dfrac{\rho}{\epsilon_0}$.

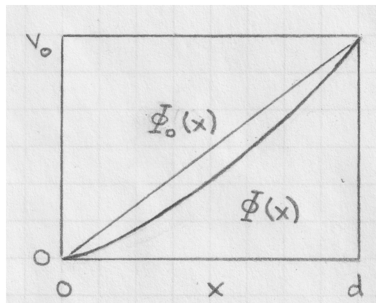
(4) Functional relation between $\rho(x)$ and $\Phi(x)$ inferred from (1) and (2) with (uniform) current density J still undetermined:

$$\rho(x) = \frac{J}{v(x)} = -\frac{c}{\sqrt{\Phi(x)}}, \quad c \doteq -J\sqrt{\frac{m}{2e}} > 0.$$

(5) ODE for $\Phi(x)$ inferred from (3) and (4) and boundary conditions:

$$\frac{d^2\Phi}{dx^2} = \frac{c}{\epsilon_0\sqrt{\Phi}}, \quad \Phi(0) = 0, \quad \Phi(d) = V_0.$$

(6) Trial function that satisfies boundary condition: $\Phi(x) = V_0 \left(\frac{x}{d}\right)^{4/3}$.



(7) The trial function of (6) satisfies ODE in (5) for a particular value of the constant c, i.e. for a particular value of the current density J:

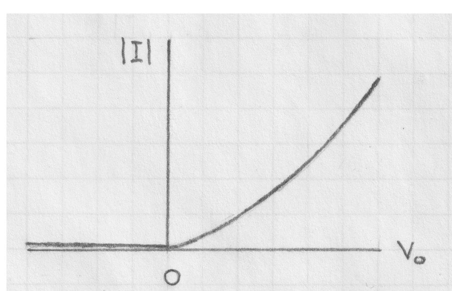
$$\frac{d\Phi}{dx} = \frac{4}{3}\frac{V_0}{d}\left(\frac{x}{d}\right)^{1/3}, \quad \frac{d^2\Phi}{dx^2} = \frac{4}{9}\frac{V_0}{d^2}\left(\frac{x}{d}\right)^{-2/3} = \underbrace{\frac{4}{9}\frac{V_0^{3/2}}{d^2}}_{c/\epsilon_0}\underbrace{V_0^{-1/2}\left(\frac{x}{d}\right)^{-2/3}}_{1/\sqrt{\Phi_0}}$$

$$\Rightarrow \; c = \frac{4\epsilon_0 V_0^{3/2}}{9d^2} \quad \Rightarrow \; J = -c\sqrt{\frac{2e}{m}} = -\sqrt{\frac{2e}{m}}\,\frac{4\epsilon_0 V_0^{3/2}}{9d^2}.$$

(8) Current-voltage characteristic of vacuum-tube diode:

$$I \doteq JA = -\sqrt{\frac{2e}{m}}\,\frac{4\epsilon_0 A}{9d^2}V_0^{3/2} \quad : \; V_0 > 0 \quad \text{(Child – Langmuir law)}.$$

Ohm's law between current and voltage is not satisfied. The resistance of the vacuum tube depends on the current. Negative voltage, $V_0 < 0$ blocks the current. No electrons are emitted from the unheated anode.

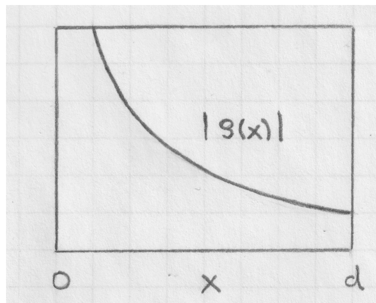


The vacuum-tube has the function of a diode (prohibition of current reversal).

(9) Electron charge density between the electrodes from (4):

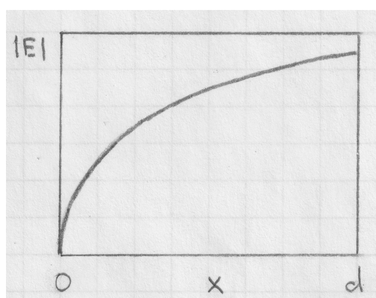
$$\rho(x) = -\frac{c}{\sqrt{\Phi(x)}} = -\frac{4\epsilon_0 V_0}{9d^2}\left(\frac{x}{d}\right)^{-2/3}.$$

The space charge density is negative and its magnitude is high near the cathode and low near the anode.



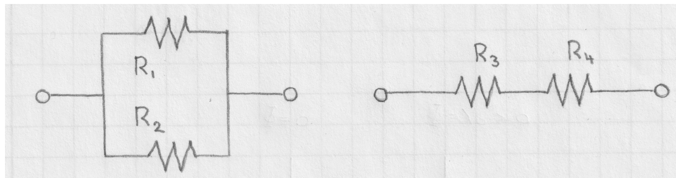
(10) The electric field is weak near the cathode and strong near the anode:

$$E = -\frac{d\Phi}{dx} = -\frac{4V_0}{3d}\left(\frac{x}{d}\right)^{1/3}.$$



Resistor circuits in steady state:

There are two distinct ways in which a pair of resistors can be connected, as a unit with two terminals, to other parts of a circuit. Such units can be replaced by a single, equivalent resistor with the same function in the circuit.



Parallel configuration:

– Current through resistors: $I = I_1 + I_2$.

– Voltage across resistors: $V = V_1 = V_2$.

– Equivalent resistance: $\dfrac{1}{R_{\mathrm{eq}}} \doteq \dfrac{I}{V} = \dfrac{I_1 + I_2}{V} = \dfrac{I_1}{V_1} + \dfrac{I_2}{V_2} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.

Series configuration:

– Current through resistors: $I = I_3 = I_4$.

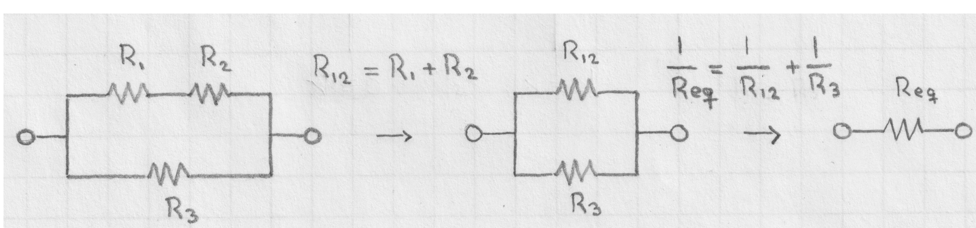
– Voltage across resistors: $V = V_3 + V_4$.

– Equivalent resistance: $R_{\mathrm{eq}} \doteq \dfrac{V}{I} = \dfrac{V_3 + V_4}{I} = \dfrac{V_3}{I_3} + \dfrac{V_4}{I_4} = R_3 + R_4$.

Every resistor circuit, no matter how complex, with a voltage V between two terminals that produces a steady current I can be reduced to one equivalent resistor with the same functions.

The complete analysis of a resistor circuit with resistances R_i requires the determination of the current I_i flowing through each resistor. The voltage across each resistor and power dissipated in each resistor then follow directly: $V_i = R_i I_i$, $P_i = R_i I_i^2$.

Some resistor circuits can be analyzed by sequentially reducing units of parallel or series configurations down to the equivalent resistor.



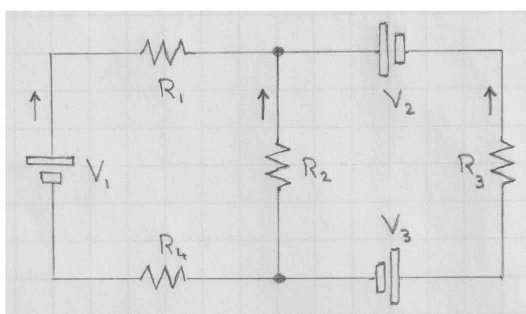
In general, the analysis of resistor circuits in steady state with one or several EMF sources is based on Kirchhoff's rules:

- *Resistor rule*: In the direction of a current I across a resistor with resistance R, there is a drop $-IR$ in electric potential.
- *EMF rule*: From the $(-)$ terminal to the $(+)$ terminal in an ideal voltage source, the potential rises by the amount V stated on its label.
- *Junction rule*: At any junction, the sum of incoming currents is equal to the sum of outgoing currents.
- *Loop rule*: The sum of changes in electric potential across devices around any closed loop is zero.

Recall that currents in a circuit are flux quantities of current densities through open surfaces across wires. The area vector for open surfaces is a matter of choice. Therefore, all current directions are a matter of choice.

Begin the analysis of a resistor circuit by naming the currents along each branch (between junctions), which includes assigning a direction.

The application of Kirchhoff's rules leads to a set of linear algebraic equations for the currents. The solution for any set of specifications is unique. Some currents may come out negative.



The circuit shown has two junctions and three branches with assigned current directions indicated.

We first name the currents and impose the junction rule (e.g. I_1, I_2, and $-I_1-I_2$ from left to right). There are two independent currents. Imposing the loop rule twice (e.g. for the loops V_1, R_1, R_2, R_4 and V_2, R_3, V_3, R_2) produces two additional linear equations for the unknown currents. [lex35][lex49]

A positive current flows in the direction shown. A negative current in the direction shown is equivalent to a positive current in the opposite direction.

Device with capacitance and resistance:

Consider two oppositely charged conducting plates with negligible resistivity connected by a battery that supplies a voltage V.

The space between the plates is filled with a material that has permittivity ϵ and conductivity σ.

Consider a steady state with the plates carrying charge $\pm Q$ and a current I flowing as shown.[2]

Use local relations: $\mathbf{J} = \sigma\mathbf{E}, \quad \nabla\cdot\mathbf{E} = \rho_\mathrm{f}/\epsilon$.

Use Gauss's theorem.

Consider the flux of current density $\mathbf{J}$ (current I) and the flux of electric field $\mathbf{E}$ (related to charge inside by Gauss's law) through closed surface S.

$$I = \oint_S d\mathbf{a}\cdot\mathbf{J} = \sigma\oint_S d\mathbf{a}\cdot\mathbf{E} = \sigma\int d^3x\,\nabla\cdot\mathbf{E} = \frac{\sigma}{\epsilon}\int d^3x\,\rho_\mathrm{f} = \frac{\sigma}{\epsilon}Q.$$

Device properties: resistance $R = d/\sigma A$, capacitance $C = \epsilon A/d$.

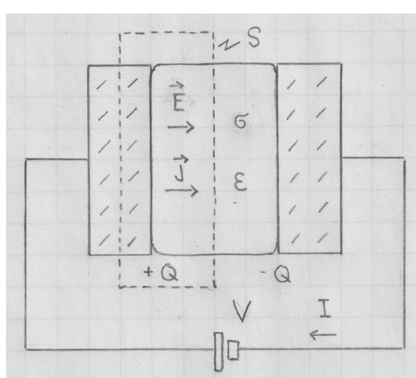
Material properties: conductivity σ, permittivity ϵ.

Device relations: $V = RI, \quad Q = CV$.

Relation between device properties and material properties:

$$\frac{Q}{I} = \frac{CV}{V/R} = RC = \frac{\epsilon}{\sigma} \doteq \tau.$$

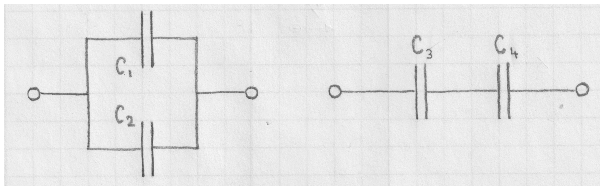
The characteristic time τ describes (i) the decay of charge concentrations on the material level and (ii) the charging/discharging processes of capacitors in RC circuits on the device level.



[2]The tacit assumption here is that the EMF source is temporarily disconnected. The only current through S happens inside the dielectric material.

Capacitor circuits at equilibrium:

There are two distinct ways in which a pair of capacitors can be connected, as a unit with two terminals, to other parts of a circuit. Such units can be replaced by a single, equivalent capacitor with the same function in the circuit.



Parallel configuration:

– Charge on capacitors: $Q = Q_1 + Q_2$.

– Voltage across capacitors: $V = V_1 = V_2$.

– Equivalent capacitance: $C_{\rm eq} \doteq \dfrac{Q}{V} = \dfrac{Q_1 + Q_2}{V} = \dfrac{Q_1}{V_1} + \dfrac{Q_2}{V_2} = C_1 + C_2$.

Series configuration:

– Charge on capacitors: $Q = Q_3 = Q_4$.

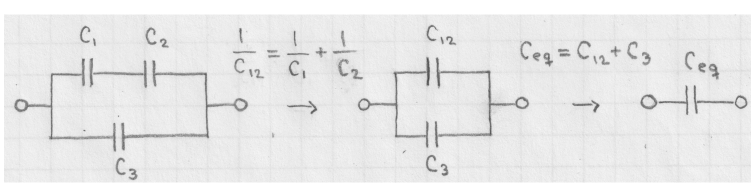
– Voltage across capacitors: $V = V_3 + V_4$.

– Equivalent capacitance: $\dfrac{1}{C_{\rm eq}} \doteq \dfrac{V}{Q} = \dfrac{V_3 + V_4}{Q} = \dfrac{V_3}{Q_3} + \dfrac{V_4}{Q_4} = \dfrac{1}{C_3} + \dfrac{1}{C_4}$.

Every capacitor circuit, no matter how complex, with a voltage V between two terminals and at equilibrium, can be reduced to one equivalent capacitor with the same functions.[3]

The complete analysis of a capacitor circuit with resistances C_i requires the determination of the charge Q_i on each capacitor. The voltage and the energy storage then follow directly: $V_i = Q_i/C_i$, $U_i = Q_i^2/2C_i$.

Some capacitor circuits can be analyzed by sequentially reducing units of parallel or series configurations down to the equivalent capacitor.



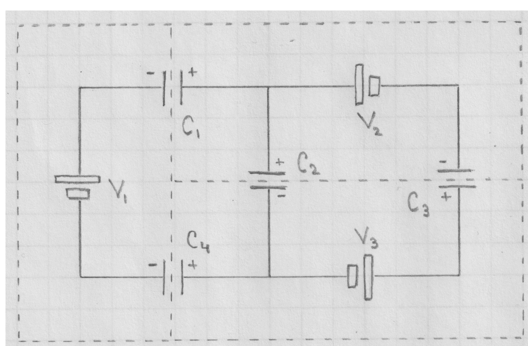
[3]The tacit assumption is that there is no charge on any capacitor before the circuit is connected to the voltage source.

In general, the analysis of capacitor circuits connected to one or several EMF sources at equilibrium is based on the following rules:

- *Capacitor rule*: From the negatively charged conductor to the positively charged conductor of a capacitor with capacitance C, the potential rises by Q/C.
- *EMF rule*: From the $(-)$ terminal to the $(+)$ terminal in an ideal voltage source, the potential rises by the amount stated on its label.
- *Conductor rule*: On any conductor, the total charge is zero. All conductors begin and end in a capacitor.
- *Loop rule*: The sum of changes in electric potential across devices around any closed loop is zero.

Begin the analysis of a capacitor circuit by assigning positive charge $+Q_i$ and negative charge $-Q_i$ to opposite conductors of each capacitor (with capacitance C_i).

The application of the four rules leads to a set of linear equations for the charges Q_i. The solution for any set of specifications is unique. Some charges may come out negative.



The circuit shown consists of three conductors delimited by dashed rectangles. All excess charge is on the capacitors with assigned signs indicated. [lex34]

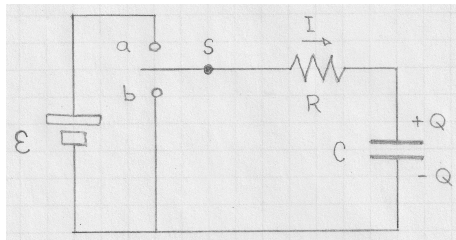
The conductor rules read $-Q_1-Q_4=0$, $Q_2+Q_1-Q_3=0$, $Q_4+Q_3-Q_2=0$. [lex55][lex56]
Only two of the three equations are independent.

Two further linear equations for the four unknowns $Q_1,\ldots,Q_4$ come from imposing the loop rule on two loops of choice.

A positive Q_1 means that the plate on the left is negatively charged and the plate on the right positively. The opposite is the case if Q_1 is negative.

RC circuits:

One-loop circuit with battery, resistor and capacitor. The switch S allows the battery to be connected and disconnected.



Specifications: EMF $\mathcal{E}$, resistance R, capacitance C.

Time-dependent quantities:

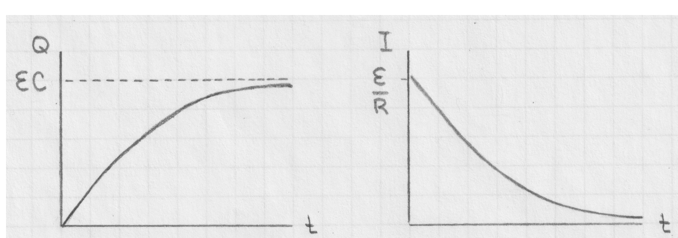
- charge on capacitor: $Q(t)$,
- current through loop: $I(t) = \dfrac{dQ}{dt}$,
- voltage across resistor: $V_R(t) = RI(t)$,
- voltage across capacitor: $V_C(t) = \dfrac{Q(t)}{C}$.

(i) *Charging process* (battery connected, $S \to a$):

- Loop rule: $\mathcal{E} - I(t)R - \dfrac{Q(t)}{C} = 0$.
- ODE solved by separation of variables: $\dfrac{dQ}{dt} = \dfrac{\mathcal{E}C - Q}{RC}$

$$\Rightarrow \int_0^Q \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC} \quad \Rightarrow \quad -\ln\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = \frac{t}{RC}.$$

- Charge: $Q(t) = \mathcal{E}C\left[1 - e^{-t/RC}\right]$.
- Current: $I(t) = \dfrac{\mathcal{E}}{R}\, e^{-t/RC} > 0$.



– Rates of energy transfer: $I(t)\mathcal{E} - I^2(t)R - \dfrac{I(t)Q(t)}{C} = 0$,

$I\mathcal{E} > 0$: rate at which battery delivers energy,

$IV_R = I^2R > 0$: rate at which resistor dissipates energy,

$IV_C = \dfrac{IQ}{C} > 0$: rate at which capacitor stores energy.

– Charging characteristics:

$\dfrac{\mathcal{E}}{R}$: rate at which current flows onto capacitor initially,

$\mathcal{E}C$: total charge placed on charge ultimately,

RC: time it takes to place ~63% of total charge onto capacitor.

(ii) *Discharging process* (battery disconnected, $S \to b$):

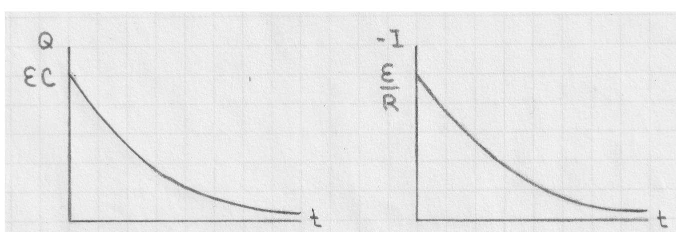
– Loop rule: $I(t)R + \dfrac{Q(t)}{C} = 0$.

– ODE solved by separation of variables: $\dfrac{dQ}{dt} = -\dfrac{Q}{RC}$

$$\Rightarrow \int_{\mathcal{E}C}^{Q} \frac{dQ}{Q} = -\int_0^t \frac{dt}{RC} \quad \Rightarrow \ln\left(\frac{Q}{\mathcal{E}C}\right) = -\frac{t}{RC}.$$

– Charge: $Q(t) = \mathcal{E}Ce^{-t/RC}$.

– Current: $I(t) = -\dfrac{\mathcal{E}}{R}e^{-t/RC} < 0$ (note reverse direction).



– Rates of energy transfer: $I^2(t)R + \dfrac{I(t)Q(t)}{C} = 0$,

$IV_R = I^2R > 0$: rate at which resistor dissipates energy,

$IV_C = \dfrac{IQ}{C} < 0$: rate at which capacitor releases energy.

- Discharging characteristics:

  $\dfrac{\mathcal{E}}{R}$: rate at which current flows off capacitor initially,

  $\mathcal{E}C$: total charge released from charge ultimately,

  RC: time it takes to release ∼63% of total charge from capacitor.

- Energy stored on capacitor initially: $U = \dfrac{1}{2}C\mathcal{E}^2$.
- The rate at which the resistor dissipates energy depends on R but the total energy dissipated in the resistor during the discharging process is independent of R:

$$\int_0^\infty dt\, I^2(t)R = \frac{\mathcal{E}^2}{R}\int_0^\infty dt\, e^{-2t/RC} = \frac{1}{2}C\mathcal{E}^2.$$

Key points to remember in the analysis of RC circuits: [lex57][lex58] [lex61][lex198]

- The voltage across a capacitor with no charge is zero. An empty capacitor is an invisible device.
- There is no steady current through a capacitor. A capacitor is charged up by a unidirectional current and then blocks any further current in the same direction.