Surface Electric Dipole Layers [lam9]

Boundary conditions of electrostatic situations often involve surface charge densities $\sigma(\mathbf{x})$. They produce discontinuities in the normal component of the electric field as discussed and used in [lln5].

Discontinuities in the electric potential $\Phi(\mathbf{x})$, not nearly as often in evidence, can be produced by double layers $+\sigma(\mathbf{x})$ and $-\sigma(\mathbf{x})$, separated by a small distance $d(\mathbf{x})$.

Surface electric dipole density: $D(\mathbf{x}) \doteq \lim_{d(\mathbf{x})\to 0} \sigma(\mathbf{x}) d(\mathbf{x})$.

Electric potential of a surface electric dipole layer:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_S da'\sigma(\mathbf{x}')\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}-\mathbf{x}'+\hat{\mathbf{n}}\, d(\mathbf{x}')|}\right] \tag{1}$$

Expand the second term: $\displaystyle \frac{1}{|\mathbf{x}+\mathbf{b}|} \overset{|\mathbf{b}|\ll|\mathbf{x}|}{=} \frac{1}{|\mathbf{x}|} + \mathbf{b}\cdot\nabla\left(\frac{1}{|\mathbf{x}|}\right)$.

Convert the integrand: $\displaystyle \hat{\mathbf{n}}\cdot\nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)da' = -\frac{\cos\theta da'}{|\mathbf{x}-\mathbf{x}'|^2} = -d\Omega$.

$$\Rightarrow \;\; \Phi(\mathbf{x}) = -\frac{1}{4\pi\epsilon_0}\int_S d\Omega\, D(\mathbf{x}'). \tag{2}$$

The position $\mathbf{x}$ of the field point is hidden in the element of solid angle $d\Omega$ in expression (2):

- Consider a flat and uniform dipole density D with two field points close by on opposite sides: $\int d\Omega = \pm 2\pi \quad \Rightarrow \;\; \Delta\Phi = D/\epsilon_0$.
- Consider a field point inside a closed surface of uniform dipole density: $\int d\Omega = 4\pi \quad \Rightarrow \;\; \Phi_{\mathrm{int}} = -D/\epsilon_0$.
- Consider a field point outside a closed surface of uniform dipole density: $\int d\Omega = 0 \quad \Rightarrow \;\; \Phi_{\mathrm{int}} = 0$.