Electrostatic Energy of Point Charges: Self-Energy and Interaction Energy [lam8]

We have noted in [lln5] that the expression,

$$U = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int d^3x \int d^3x' \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2}\int d^3x \rho(\mathbf{x})\Phi(\mathbf{x}), \tag{1}$$

for the potential energy of a charge distribution $\rho(\mathbf{x})$ is not equivalent to the expression,

$$U_{\text{int}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\mathbf{x}_i-\mathbf{x}_j|}, \tag{2}$$

for the potential interaction energy of an array of discrete charges q_i. The former expression converted into an integral over energy density $w(\mathbf{x})$ as demonstrated in [lln5],

$$U = \frac{1}{2}\int d^3x \rho(\mathbf{x})\Phi(\mathbf{x}) = \frac{1}{2}\epsilon_0 \int d^3x |\mathbf{E}(\mathbf{x})|^2 = \int d^3x\, w(\mathbf{x}), \tag{3}$$

is manifestly non-negative, whereas the latter expression can be negative (e.g. for two point charges of opposite sign).

We have also shown in [lln5] how the the integral expressions in (3) can be split into the sum of self-energy and interaction energy.

The energy density generated by the electric field of two point charges is

$$\begin{aligned} w(\mathbf{x}) &= \frac{1}{2}\epsilon_0\left[\frac{q_1}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_1}{|\mathbf{x}-\mathbf{x}_1|^3} + \frac{q_2}{4\pi\epsilon_0}\frac{\mathbf{x}-\mathbf{x}_2}{|\mathbf{x}-\mathbf{x}_2|^3}\right]^2 \\ &= \frac{1}{32\pi^2\epsilon_0}\left[\frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + \frac{q_1^2}{|\mathbf{x}-\mathbf{x}_1|^4} + 2q_1q_2\,\frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3}\right]. \end{aligned} \tag{4}$$

The first two terms represent, upon integration, the electrostatic self-energies, which are divergent and not included in U_{int} as constructed in (2).

Divergent charge densities or energy densities do not, in general, produce infinite self-energies. The problem here is caused by having finite charges concentrated in points, which yields strong (non-integrable) divergences.

The last term in (4) represents U_{int} and can be transformed (upon integration) as follows [lex120]:

$$U_{\text{int}} = \frac{q_1q_2}{16\pi^2\epsilon_0}\int d^3x \frac{(\mathbf{x}-\mathbf{x}_1)\cdot(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_1|^3|\mathbf{x}-\mathbf{x}_2|^3} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_1}{|\mathbf{x}_1-\mathbf{x}_2|}.$$