Quantum Harmonic Oscillator [lam5]

Here we summarize the quantum mechanical treatment of the linear harmonic oscillator, i.e. in a one-dimensional configuration space.

Hamiltonian: $\mathcal{H} = \frac{1}{2m}\mathcal{P}^2 + \frac{1}{2}m\omega^2\mathcal{Q}^2$.

Hermitian operators: $\mathcal{H},\ \mathcal{P},\ \mathcal{Q},\ \mathcal{I}$.

Fundamental commutator: $[\mathcal{Q},\mathcal{P}] = \imath\hbar\mathcal{I}$.

Position space:

Wave function of stationary state: $\psi(x)$.

Inner product: $\langle\psi|\psi'\rangle = \int_{-\infty}^{+\infty} dx\psi^*(x)\psi'(x) = \langle\psi'|\psi\rangle^*$.

Normalization: $\langle\psi|\psi\rangle = \int_{-\infty}^{+\infty} dx\psi^*(x)\psi(x)=1$.

Operators: $\mathcal{Q} = x, \quad \mathcal{P} = -\imath\hbar\frac{d}{dx}, \quad \mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$.

Eigenvalue equation: $\mathcal{H}\psi(x) = E\psi(x)$.

Energy levels: $E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 1, 2, \ldots$

Eigenfunctions: $\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)\mathrm{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$.

Hermite polynomials: $\mathrm{H}_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$.

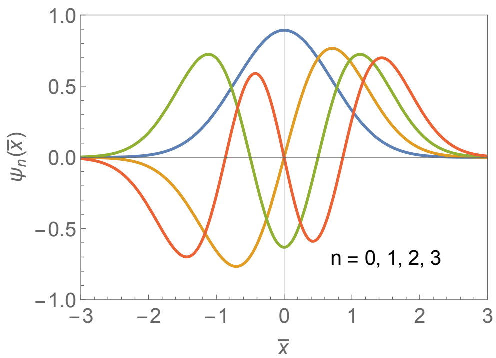
Ground state: $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$.

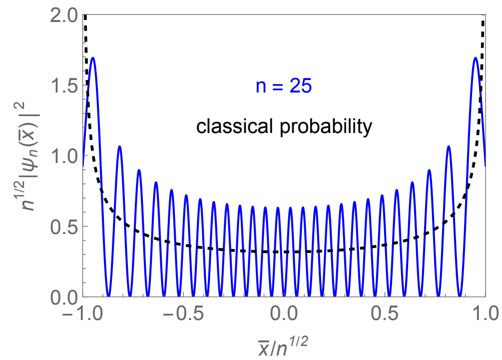
Scaled position variable: $\bar{x} \doteq \frac{x}{\sqrt{2\hbar/m\omega}}$.

The wave functions are symmetric for even n and antisymmetric for odd n. They have parity $(-1)^n$.

The probability distribution for position approaches the classical prediction in an average sense as $n \to \infty$ if subjected to appropriate rescaling.

The quantum result is strongly oscillating. Convergence toward the classical result is manifest when averaged between successive maxima or minima.





Classical probability density for the scaled position $\hat{x}(t) \doteq x(t)/x_0$ [lex182]:

$$P_x(\hat{x}) = \frac{2}{\tau}\int_{-\tau/4}^{+\tau/4} dt\, \delta\big(\hat{x} - \sin(\omega t)\big) = \frac{1}{\pi}\frac{1}{\sqrt{1-\hat{x}^2}}, \quad \tau = \frac{2\pi}{\omega}. \tag{1}$$

Momentum space:

Wave function of stationary state: $\tilde{\psi}(p)$.

Operators: $\mathcal{Q} = \imath\hbar\frac{d}{dp}, \quad \mathcal{P} = p, \quad \mathcal{H} = \frac{1}{2m}p^2 - \frac{1}{2}m\omega^2\frac{d^2}{dp^2}$.

Parameter substitution: $m\omega = \frac{1}{\mu\omega}$.

Equivalent Hamiltonian: $\mathcal{H} = -\frac{\hbar^2}{2\mu}\frac{d^2}{dp^2} + \frac{1}{2}\mu\omega^2 p^2$.

Eigenfunctions: $\tilde{\psi}_n(p) = \left(\frac{1}{\pi m\hbar\omega}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\exp\left(-\frac{p^2}{2m\hbar\omega}\right)\mathrm{H}_n\left(\frac{p}{\sqrt{m\hbar\omega}}\right)$.

Scaled momentum variable: $\bar{p} \doteq \frac{p}{\sqrt{2m\hbar\omega}}$.

Classical probability density for the scaled momentum $\hat{p}(t) \doteq p(t)/p_0$:

$$P_p(\hat{p}) = \frac{2}{\tau}\int_0^{\tau/2} dt\, \delta\big(\hat{p} - \cos(\omega t)\big) = \frac{1}{\pi}\frac{1}{\sqrt{1-\hat{p}^2}}, \quad \tau = \frac{2\pi}{\omega}. \tag{2}$$

The graphs in momentum space and position space are identical for appropriately scaled quantities.

The probability densities $P_x(\hat{x})$ and $P_p(\hat{p})$ are constructed from classical time averages to match quantum averages of stationary states.

The variables $\hat{x}$ and $\hat{p}$ are not statistically independent. Hence the probability densities $P_x(\hat{x})$ and $P_p(\hat{p})$ do not factorize.

Classical states in phase space:

The (deterministic) classical time evolution of the scaled variables $\hat{x}(t) = \sin(\omega t)$ and $\hat{p} = \cos(\omega t)$ are represented in phase space by a point moving at constant angular velocity ω in a clockwise sense.

Joint probability distribution associated with the phase point in motion:

$$P(\hat{x}, \hat{p}, t) = \delta\big(\hat{x} - \sin(\omega t)\big)\delta\big(\hat{p} - \cos(\omega t)\big). \tag{3}$$

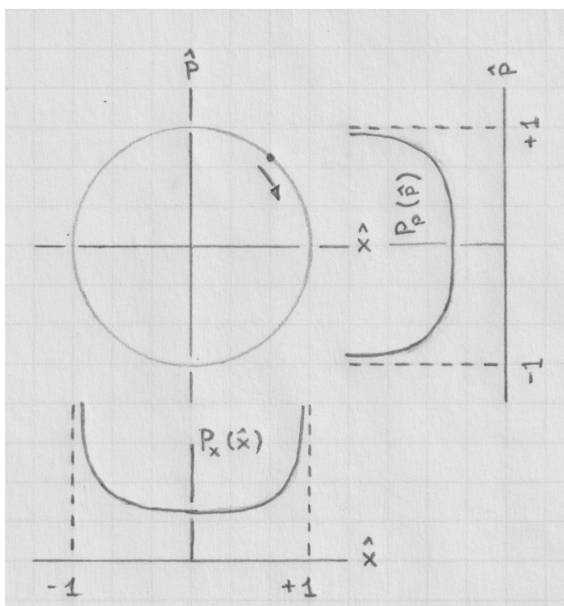
The probability densities $P_x(\hat{x})$ and $P_p(\hat{p})$ are recovered from this expression by first integrating over $\hat{p}$ or $\hat{x}$, respectively, and the taking the time average.

Taking only the time average yields the stationary phase-space distribution,

$$\bar{P}(\hat{x}, \hat{p}) = \frac{1}{2\pi}\delta\Big(\sqrt{\hat{x}^2 + \hat{p}^2} - 1\Big). \tag{4}$$

Here probability is uniformly distributed over the unit circle, which represents what is called an invariant torus in Hamiltonian dynamics.

The probability densities $P_x(\hat{x})$ and $P_p(\hat{p})$ can also be recovered recovered from this expression, by integrating over $\hat{p}$ or $\hat{x}$, respectively [lex184].



We have already connected the classical probability densities (1) and (2) with quantum wave functions in position space and momentum space, respectively.

Equivalent connections of classical probability densities (3) and (4) with quantum states remain to be established.

Number states:

Ladder operators (raising/lowering, creation/annihilation):

$$a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\big(m\omega\mathcal{Q} - \imath\mathcal{P}\big), \quad a = \frac{1}{\sqrt{2m\hbar\omega}}\big(m\omega\mathcal{Q} + \imath\mathcal{P}\big).$$

$$\Rightarrow\ \mathcal{P} = \imath\sqrt{\frac{m\hbar\omega}{2}}\big(a^\dagger - a\big), \quad \mathcal{Q} = \sqrt{\frac{\hbar}{2m\omega}}\big(a + a^\dagger\big).$$

The creation and annihilation refers to bosonic particles.

Fundamental commutator: $[a, a^\dagger] = 1$.

Hamiltonian: $\mathcal{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$.

Number operator: $\mathcal{N} = a^\dagger a$.

Commuting operators: $[\mathcal{H}, \mathcal{N}] = 0$.

Number states: $|n\rangle, \quad n = 0, 1, 2, \ldots$ with $\langle n|n'\rangle = \delta_{nn'}$.

Action of ladder and number operators:

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle \quad \Rightarrow\ a^\dagger a|n\rangle = n|n\rangle.$$

Energy levels: $E_n = \langle n|\mathcal{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$

Generation of number-state eigenvectors from ground state:

$$|1\rangle = a^\dagger|0\rangle, \quad |2\rangle = \frac{1}{\sqrt{2}}a^\dagger|1\rangle = \frac{1}{\sqrt{2!}}(a^\dagger)^2|0\rangle, \ldots$$

$$\Rightarrow\ |n\rangle = \frac{1}{\sqrt{n}}a^\dagger|n-1\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle.$$

The state $|0\rangle$ is the ground state (physical vacuum) and the state with $n = 0$ particles (pseudo-vacuum). The two vacua do not, in general, coincide.

Time evolution of number states,

$$\imath\hbar\frac{d}{dt}|n\rangle = \mathcal{H}|n\rangle = \hbar\omega_n|n\rangle, \quad \omega_n = \left(n + \frac{1}{2}\right)\omega \quad \Rightarrow\ |n(t)\rangle = |n\rangle e^{-\imath\omega_n t},$$

qualifies them as stationary states. The evidence will be presented below.

Coherent states:

Construction of coherent states $|\alpha\rangle$, where α is a complex-valued continuous parameter, from number states $|n\rangle$:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

Coherent states are normalized, but do not form an orthonormal set [lex185]:

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}.$$

The overcompleteness of coherent states is evident in the relations [lex185],

$$\int d^2\beta \, |\langle\beta|\alpha\rangle|^2 = \pi, \quad \mathcal{I} = \frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha|.$$

Coherent states are eigenstates of ladder operators [lex154]:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \langle\alpha|a^\dagger = \langle\alpha|\alpha^*.$$

The number operator has equal mean and variance in coherent states [lex154]:

$$\langle\alpha|\mathcal{N}|\alpha\rangle = |\alpha|^2, \quad \langle\alpha|\mathcal{N}^2|\alpha\rangle - \langle\alpha|\mathcal{N}|\alpha\rangle^2 = |\alpha|^2.$$

The overlap of number states with coherent states is characterized by the Poisson distribution [lex154]:

$$P(n) \doteq |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}.$$

The average values of the position and momentum operators in a coherent state are determined by the real and imaginary parts of its complex parameter [lex185]:

$$\langle\alpha|\mathcal{Q}|\alpha\rangle \doteq x = \sqrt{\frac{2\hbar}{m\omega}} \, \Re[\alpha], \quad \langle\alpha|\mathcal{P}|\alpha\rangle \doteq p = \sqrt{2m\hbar\omega} \, \Im[\alpha].$$

Calculating the uncertainties of the position and momentum operators in a coherent state produces the same results as obtained for the ground state [lex183] – the minimum value permitted by Heisenberg's inequality [lex185]:

$$\Delta Q \Delta P = \frac{1}{2}\hbar.$$

This minimum uncertainty is a consequence of the fact that coherent states can be generated from the ground state by the action of a displacement operator [lex155]:

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad D(\alpha) = e^{\imath(p\mathcal{Q}-x\mathcal{P})/\hbar} = e^{\alpha a^\dagger - \alpha * a}.$$

The time evolution of coherent states as inferred from Schrödinger equation and the time evolution of number states established earlier, is complex and makes them non-stationary:

$$|\alpha(t)\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\alpha_0^n}{\sqrt{n!}}|n\rangle e^{-\imath(n+1/2)\omega t} = \left[e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{[\alpha_0 e^{-\imath\omega t}]^n}{\sqrt{n!}}\right]|n\rangle e^{-\imath\omega t/2}.$$