Uncertainty [lam4]

The concept of uncertainty comes into play in quantum optics at two levels, in the forms of (more general) statistical uncertainty and (more specific) quantum uncertainty.

Statistical uncertainty and information:

For an experiment that has n possible outcomes with probabilities $P_1, \ldots, P_n$, any quantitative measure of uncertainty must satisfy five criteria:[1]

- The uncertainty is a function $\Sigma(P_1, P_2, \ldots, P_n)$, $\quad \sum_{i=1}^{n} P_i = 1$.
- The uncertainty is symmetric under permutations of the P_i.
- The uncertainty is highest if all P_i are equal.
- The uncertainty is zero if one outcome has probability $P_i = 1$.
- The combined uncertainty for two independent experiments is the sum of their uncertainties:

$$P_{ij} = P_i^{\mathrm{a}} P_j^{\mathrm{b}} \quad \Rightarrow \Sigma(\{P_{ij}\}) = \Sigma_{\mathrm{a}}(\{P_i^{\mathrm{a}}\}) + \Sigma_{\mathrm{b}}(\{P_j^{\mathrm{b}}\}).$$

The following function satisfies all criteria:

$$\Sigma(P_1, P_2, \ldots, P_n) = -\sum_{i=1}^{n} P_i \ln P_i = -\langle \ln P \rangle. \tag{1}$$

We have used this expression in [lln24] to compare the uncertainty associated with photon detection in light of super-Poisson statistics.

Information (in a restricted sense) is carried by messages. The information content $I(M)$ of a message M is tied to the probability $P(M)$ that the message is being received. Applicable criteria are the following:

- If $P(A) < P(B)$ then $I(A) > I(B)$.
- If $P(A) = 1$ then $I(A) = 0$.
- If $P(A \cap B) = P(A)P(B)$ then $I(A \cap B) = I(A) + I(B)$.

The information content (in the statistical sense) of a message is equal to the change in statistical uncertainty at the receiver:

$$P_1, P_2, \ldots, P_n \xrightarrow{A} \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n$$
$$\Rightarrow \; I(A) = \Sigma(P_1, P_2, \ldots, P_n) - \Sigma(\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n).$$

[1] These criteria must be modified if the probabilities are subject to auxiliary conditions.

Statistical uncertainty and entropy:

From [lam1] we know that the state of a quantum system can be specified, quite generally, by a density operator,

$$\rho = \sum_\psi P_\psi |\psi\rangle\langle\psi|, \quad \sum_\psi P_\psi = 1. \tag{2}$$

Pure quantum states (not mixed states) are free of statistical uncertainty as expressed by (1). Statistical mechanics connects (pure) microstates with (mixed) macrostates. The latter are considered in thermodynamics.

The microstate of a macroscopic system encodes the maximum information attainable about it in principle – the least uncertainty about it.

The function (1) with the P_n from (2) is a measure for the uncertainty about the microstate if the system is in a specific macrostate.

A macroscopic system in isolation settles down (at equilibrium) in a state of maximum entropy. While it approaches equilibrium, the entropy increases as does our uncertainty about the microstate.

The relation between density operator and entropy as established in statistical mechanics is

$$S = -k_\mathrm{B}\langle \ln P\rangle = -k_\mathrm{B}\sum_\psi P_\psi \ln P_\psi.$$

Associating entropy with uncertainty about the microstate makes sense epistemologically. From this perspective, thermodynamic equilibrium is the macrostate that makes it hardest to predict the actual microstate.

We shall see next that even microstates (pure quantum states) are subject to a form of uncertainty, namely quantum uncertainty.

Heisenberg uncertainty principle:

Two commuting operators have a common set of eigenvectors. There exist pure quantum states in which the expectation values of both operators are free of uncertainty.

The expectation values of two non-commuting operators, $[A, B] \neq 0$, in every pure quantum state are subject to uncertainty in the sense that the product of their standard deviations has a minimum value:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|, \quad \Delta A \doteq \sqrt{\langle A^2\rangle - \langle A\rangle^2}, \quad \Delta B \doteq \sqrt{\langle B^2\rangle - \langle B\rangle^2}.$$

For quantized canonically conjugate coordinates, we have $\Delta Q \Delta P = \frac{1}{2}\hbar$.