Quantization of the Electromagnetic Field [lam3]

The purpose of this elementary introduction is tailored toward the needs of quantum optics. QED will later be introduced more systematically.

We consider a region of free space in the form of a cube of volume $V = L^3$ with periodic boundary conditions imposed. There are no sources present: $\rho \equiv 0$, $\mathbf{J} \equiv 0$.

From [lln15] we know that the (dynamic) electric and magnetic fields can by derived from a vector potential $\mathbf{A}(\mathbf{r}, t)$, which satisfies the wave equation,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \quad c = \frac{1}{\epsilon_0 \mu_0},$$

via derivatives as follows:

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}.$$

The electromagnetic field energy is

$$U = \int_V d^3r \left[\frac{1}{2}\epsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{2\mu_0}\mathbf{B}^2(\mathbf{r}, t)\right].$$

The wave equation is linear and of 2^{nd} order. The variables $\mathbf{r}$ and t can be separated by a product ansatz.

The general solution can be (Fourier) expanded into orthonormal transverse plane-wave solutions:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}\lambda} \left[\bar{A}_{\mathbf{k}\lambda}(t)\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) + \bar{A}^*_{\mathbf{k}\lambda}(t)\mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r})\right],$$

– spatial part of vector potential: $\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) = \hat{\mathbf{e}}_{\mathbf{k}\lambda} e^{\imath \mathbf{k}\cdot\mathbf{r}}$,

– wave vector: $\mathbf{k} = (k_x, k_y, k_z) = \dfrac{2\pi}{L}(n_x, n_y, n_z), \quad n_i = \pm 1, \pm 2, \ldots,$

– polarization vectors: $\mathbf{k} \cdot \hat{\mathbf{e}}_{\mathbf{k}\lambda} = 0, \quad \hat{\mathbf{e}}_{\mathbf{k}\lambda} \cdot \hat{\mathbf{e}}_{\mathbf{k}\lambda'} = \delta_{\lambda,\lambda'}, \quad \lambda = 1, 2,$

– right-handed triad: $\hat{\mathbf{e}}_{\mathbf{k}1}, \quad \hat{\mathbf{e}}_{\mathbf{k}2}, \quad \hat{\mathbf{k}} \doteq \mathbf{k}/k,$

– spatial components $\hat{e}^i_{\mathbf{k}\lambda}, \quad i = x, y, z,$

– relation between components: $\displaystyle\sum_\lambda \hat{e}^i_{\mathbf{k}\lambda}\hat{e}^j_{\mathbf{k}\lambda} = \delta_{ij} - \frac{k_i k_j}{k^2},$

– orthonormality: $\displaystyle\frac{1}{V}\int_V d^3r\, \mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r}) \cdot \mathbf{u}_{\mathbf{k}\lambda'}(\mathbf{r}) = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda\lambda'},$

– ODE for temporal part: $\left[c^2k^2 - \frac{d^2}{dt^2}\right]\bar{A}_{\mathbf{k}\lambda}(t) = 0,$

– solution of ODE: $\bar{A}_{\mathbf{k}\lambda}(t) = A_{\mathbf{k}\lambda}e^{-\imath\omega_k t}, \quad \omega_k = ck,$

The vector potential $\mathbf{A}(\mathbf{r},t)$ is real by construction as will be the electric and magnetic fields inferred via temporal and spatial derivatives, respectively:

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t) = \imath\sum_{\mathbf{k}\lambda}\omega_k\Big[\bar{A}_{\mathbf{k}\lambda}(t)\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) - \bar{A}^*_{\mathbf{k}\lambda}(t)\mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r})\Big],$$

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t) = \imath\sum_{\mathbf{k}\lambda}\mathbf{k}\times\Big[\bar{A}_{\mathbf{k}\lambda}(t)\mathbf{u}_{\mathbf{k}\lambda}(\mathbf{r}) - \bar{A}^*_{\mathbf{k}\lambda}(t)\mathbf{u}^*_{\mathbf{k}\lambda}(\mathbf{r})\Big].$$

Electromagnetic field energy is conserved:

$$\begin{aligned} U &= \epsilon_0 V\sum_{\mathbf{k}\lambda}\omega_k^2\Big[\bar{A}_{\mathbf{k}\lambda}(t)\bar{A}^*_{\mathbf{k}\lambda}(t) + \bar{A}^*_{\mathbf{k}\lambda}(t)\bar{A}_{\mathbf{k}\lambda}(t)\Big] \\ &= \epsilon_0 V\sum_{\mathbf{k}\lambda}\omega_k^2\Big[A_{\mathbf{k}\lambda}A^*_{\mathbf{k}\lambda} + A^*_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda}\Big]. \end{aligned}$$

Transformation (in two steps) of Fourier amplitudes into real variables:

$$A_{\mathbf{k}\lambda} = \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\alpha_{\mathbf{k}\lambda},$$

$$q_{\mathbf{k}\lambda} \doteq \sqrt{\frac{\hbar}{2\omega_k}}\big(\alpha_{\mathbf{k}\lambda} + \alpha^*_{\mathbf{k}\lambda}\big), \quad p_{\mathbf{k}\lambda} \doteq -\imath\sqrt{\frac{\hbar\omega_k}{2}}\big(\alpha_{\mathbf{k}\lambda} - \alpha^*_{\mathbf{k}\lambda}\big).$$

Transformed electromagnetic energy:

$$U = \frac{1}{2}\sum_{\mathbf{k}\lambda}\hbar\omega_k\big(\alpha_{\mathbf{k}\lambda}\alpha^*_{\mathbf{k}\lambda} + \alpha^*_{\mathbf{k}\lambda}\alpha_{\mathbf{k}\lambda}\big) = \frac{1}{2}\sum_{\mathbf{k}\lambda}\big(p^2_{\mathbf{k}\lambda} + \omega_k^2 q^2_{\mathbf{k}\lambda}\big).$$

Plane-wave solutions are equivalent to independent modes of harmonic oscillations. The real variables $q_{\mathbf{k}\lambda}$, $p_{\mathbf{k}\lambda}$ are canonical variables.

The quantization of canonical variables is a standard affair. The $q_{\mathbf{k}\lambda}$, $p_{\mathbf{k}\lambda}$ are replaced by Hermitian operators, $Q_{\mathbf{k}\lambda}$, $P_{\mathbf{k}\lambda}$, with commutation rules,

$$[Q_{\mathbf{k}\lambda}, P_{\mathbf{k}'\lambda'}] = \imath\hbar\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}, \quad [Q_{\mathbf{k}\lambda}, Q_{\mathbf{k}'\lambda'}] = [P_{\mathbf{k}\lambda}, P_{\mathbf{k}'\lambda'}] = 0.$$

The quantization of the transformed Fourier amplitudes $\alpha_{\mathbf{k}\lambda}$, $\alpha^*_{\mathbf{k}\lambda}$ produces boson creation and annihilation operators $a_{\mathbf{k}\lambda}$, $a^\dagger_{\mathbf{k}\lambda}$:

$$Q_{\mathbf{k}\lambda} \doteq \sqrt{\frac{\hbar}{2\omega_k}}\big(a_{\mathbf{k}\lambda} + a^\dagger_{\mathbf{k}\lambda}\big), \quad P_{\mathbf{k}\lambda} \doteq -\imath\sqrt{\frac{\hbar\omega_k}{2}}\big(a_{\mathbf{k}\lambda} - a^\dagger_{\mathbf{k}\lambda}\big),$$

$$a_{\mathbf{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega_k}}\big(\omega_k Q_{\mathbf{k}\lambda} + \imath P_{\mathbf{k}\lambda}\big), \quad a^\dagger_{\mathbf{k}\lambda} = \frac{1}{\sqrt{2\hbar\omega_k}}\big(\omega_k Q_{\mathbf{k}\lambda} - \imath P_{\mathbf{k}\lambda}\big).$$

The commutation rules for boson ladder operators are:

$$[a_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}, \quad [a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}] = [a^\dagger_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}'\lambda'}] = 0.$$

Number operator: $\mathcal{N}_{\mathbf{k}\lambda} \doteq a^\dagger_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}$.

$$\Rightarrow \; [a_{\mathbf{k}\lambda}, \mathcal{N}_{\mathbf{k}'\lambda'}] = a_{\mathbf{k}\lambda}\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}, \quad [a^\dagger_{\mathbf{k}\lambda}, \mathcal{N}_{\mathbf{k}'\lambda'}] = -a^\dagger_{\mathbf{k}\lambda}\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}.$$

Hamiltonian of the quantized electromagnetic field:

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}\lambda} \hbar\omega_k \big(a_{\mathbf{k}\lambda}a^\dagger_{\mathbf{k}\lambda} + a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}\big) = \sum_{\mathbf{k}\lambda} \hbar\omega_k \left(a^\dagger_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda} + \frac{1}{2}\right).$$

Photon number states $|n_{\mathbf{k}\lambda}\rangle$ of any given mode are eigenvectors of the quantum harmonic oscillator:

$$\mathcal{N}_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = n_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle, \quad n_{\mathbf{k}\lambda} = 0, 1, 2, \ldots$$

Action of ladder operators:

$$a^\dagger_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}+1}\,|n_{\mathbf{k}\lambda}+1\rangle, \quad a_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = \sqrt{n_{\mathbf{k}\lambda}}\,|n_{\mathbf{k}\lambda}-1\rangle.$$

Spectrum generated from the ground state $|0\rangle$ (physical vacuum):

$$|n_{\mathbf{k}\lambda}\rangle = \frac{(a^\dagger_{\mathbf{k}\lambda})^{n_{\mathbf{k}\lambda}}}{\sqrt{n_{\mathbf{k}\lambda}!}}\,|0\rangle.$$

Quantized expressions for the vector potential and the fields:

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k}\lambda} \hat{\mathbf{e}}_{\mathbf{k}\lambda}\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\Big[a_{\mathbf{k}\lambda}e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + a^\dagger_{\mathbf{k}\lambda}e^{-\imath(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\Big],$$

$$\mathbf{E}(\mathbf{r},t) = \imath\sum_{\mathbf{k}\lambda} \hat{\mathbf{e}}_{\mathbf{k}\lambda}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\Big[a_{\mathbf{k}\lambda}e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - a^\dagger_{\mathbf{k}\lambda}e^{-\imath(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\Big],$$

$$\mathbf{B}(\mathbf{r},t) = \imath\sum_{\mathbf{k}\lambda} \frac{\hat{\mathbf{k}}\times\hat{\mathbf{e}}_{\mathbf{k}\lambda}}{c}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\Big[a_{\mathbf{k}\lambda}e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - a^\dagger_{\mathbf{k}\lambda}e^{-\imath(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\Big].$$

Each term in the sum represents a single plane-wave mode. The two terms of the electric field in each mode are adjoints of each other, which makes the sum Hermitian:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^{(+)}(\mathbf{r},t) + \mathbf{E}^{(-)}(\mathbf{r},t), \quad \mathbf{E}^{(+)}(\mathbf{r},t) = \mathbf{E}^{(-)}(\mathbf{r},t)^\dagger.$$

Construction of photon number states in the form of tensor products:

$$|\{n_{\mathbf{k}\lambda}\}\rangle = |n_{\mathbf{k}_1\lambda_1}\rangle \otimes |n_{\mathbf{k}_2\lambda_2}\rangle \otimes \cdots$$

– This construction is named occupation number representation. It is also known as second quantization.

– The quantized electromagnetic field consists of an infinite set of oscillators, parametrized by $\mathbf{k}$ and λ.

– The set of oscillators is infinite even if the electromagnetic field is confined to a finite volume V and irrespective of the boundary conditions. This is due to the fact that the field exists in a continuous space.

– The eigenvectors $|\{n_{\mathbf{k}\lambda}\}\rangle$ are number states of photons with given momentum and energy (encoded in $\mathbf{k}$) and polarization (encoded in λ).

– The expectation value $\langle \mathcal{N}_{\mathbf{k}\lambda}\rangle$ counts, in general, the average number of photons in a quantized mode of the electromagnetic field. For number states that number is not subject to uncertainty: $\langle n_{\mathbf{k}\lambda}|\mathcal{N}_{\mathbf{k}\lambda}|n_{\mathbf{k}\lambda}\rangle = n_{\mathbf{k}\lambda}$.

– The expectation values of the electric and magnetic fields vanish for number states:

$$\langle n_{\mathbf{k}\lambda}|\mathbf{E}(\mathbf{r},t)|n_{\mathbf{k}\lambda}\rangle = \langle n_{\mathbf{k}\lambda}|\mathbf{B}(\mathbf{r},t)|n_{\mathbf{k}\lambda}\rangle = 0.$$

– Even if all modes are in the ground state, where $\langle \mathcal{N}_{\mathbf{k}\lambda}\rangle = 0$, the total electromagnetic energy in a finite volume is infinite:

$$\mathcal{H}|0\rangle = \frac{1}{2}\sum_{\mathbf{k}\lambda} \hbar\omega_k|0\rangle.$$

– Modes are distinguishable and infinite in numbers.

– Photons of one mode are indistinguishable. Their number can be any non-negative integer.

– Photons are massless spin-1 bosons. The n-photon wave function of any mode is symmetric under the permutation of any two photons.

– The symmetry attribute is implied in the occupation number representation by the unrestricted occupancy of photons in each mode.

– Limit of unrestricted space, $V = L^3 \to \infty$:

$$\sum_{\mathbf{k}\lambda} \longrightarrow 2\left(\frac{L}{2\pi}\right)^3 \int d^3k, \quad d^3k = 4\pi k^2 dk = \frac{4\pi}{c^3}\omega_k^2 d\omega^k.$$

– Density of modes [lln24]: $\displaystyle \frac{1}{V}\frac{dN}{d\omega_k} = \frac{2}{V}\left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{c^3}\omega_k^2 = \frac{\omega_k^2}{\pi^2 c^3}.$