Alternating Current Circuits [lam28]

Here we investigate the response of resistors, capacitors, and inductors in isolation and in combinations when driven by an ac voltage source,[1]

$$\mathcal{E}(t) = V_{\mathcal{E}} e^{\imath\omega t}. \tag{1}$$

The response of these devices is linear. The (steady-state)[2] current through the source and through each device is proportional to the amplitude $V_{\mathcal{E}}$ of the driving voltage. The angular frequency ω is unchanged.

The current through the voltage source is, generally, subject to a phase shift:

$$I(t) = I_{\mathcal{E}} e^{\imath(\omega t - \delta_{\mathcal{E}})}. \tag{2}$$

The primary task for a given circuit is to determine the current amplitude $I_{\mathcal{E}}$ and the phase shift $\delta_{\mathcal{E}}$ in this expression. Both quantities are, in general, functions of the angular frequency ω and the device specifications R, L, C.

The impedance of the circuit is the time-independent complex quantity,[3]

$$Z \doteq \frac{\mathcal{E}(t)}{I(t)} = |Z| e^{\imath\delta_{\mathcal{E}}}, \quad |Z| = \frac{V_{\mathcal{E}}}{I_{\mathcal{E}}}. \tag{3}$$

Further quantities of interests are the voltage across and the current through individual devices – functions with the same general structure.

Single-device circuits:

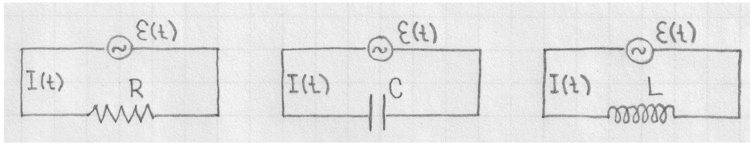
Resistor circuit:

- Source and device: $V_{\mathcal{E}} = V_R, \quad I_{\mathcal{E}} = I_R, \quad \delta_{\mathcal{E}} = \delta_R.$
- Ohm's law: $\mathcal{E}(t) = RI(t) \quad \Rightarrow \; V_R e^{\imath\omega t} = RI_R e^{\imath(\omega t - \delta_R)}.$
- Phase angle: $\delta_R = 0.$
- Impedance: $Z = \dfrac{V_R}{I_R} = R \doteq X_R \quad$ (resistance).

[1] Given that voltages and currents are real quantities, the understanding is that at the end of the analysis the real part of each quantitiy is identified as physically relevant. The real part must be taken before any nonlinear operation.

[2] The resistance in the wires – even when negligible for the quantities of interest here – guarantee that transients will fade away in due course.

[3] All impedances have SI unit [Ω]. The magnitude $|Z|$ is often named impedance as well.



Capacitor circuit:

– Source and device: $V_\mathcal{E} = V_C, \quad I_\mathcal{E} = I_C, \quad \delta_\mathcal{E} = \delta_C.$

– Charge and voltage: $\mathcal{E}(t) = V_C e^{\imath\omega t} = \dfrac{Q(t)}{C}.$

– Charge and current:

$$I(t) = \frac{dQ}{dt} = \imath\omega C V_C e^{\imath\omega t} = \omega C V_C e^{\imath(\omega t + \pi/2)} = I_C e^{\imath(\omega t - \delta_C)}.$$

– Phase angle: $\delta_C = -\dfrac{\pi}{2}.$

– Impedance: $Z = \dfrac{V_C}{I_C}\, e^{\imath\delta_C} = \dfrac{1}{\imath\omega C} \doteq X_C \quad$ (capacitive reactance).

Inductor circuit:

– Source and device: $V_\mathcal{E} = V_L, \quad I_\mathcal{E} = I_L, \quad \delta_\mathcal{E} = \delta_L.$

– Current and voltage:

$$\mathcal{E}(t) = V_L e^{\imath\omega t} = L\frac{dI}{dt} = \imath\omega L\, \underbrace{I_L e^{\imath(\omega t - \delta_L)}}_{I(t)} = \omega L I_L e^{\imath(\omega t - \delta_L + \pi/2)}.$$

– Phase angle: $\delta_L = \dfrac{\pi}{2}.$

– Impedance: $Z = \dfrac{V_L}{I_L}\, e^{\imath\delta_L} = \imath\omega L \doteq X_L \quad$ (inductive reactance).

RLC series circuit:

In this circuit, the current (2) through the ac source is the same as the current through the each device:

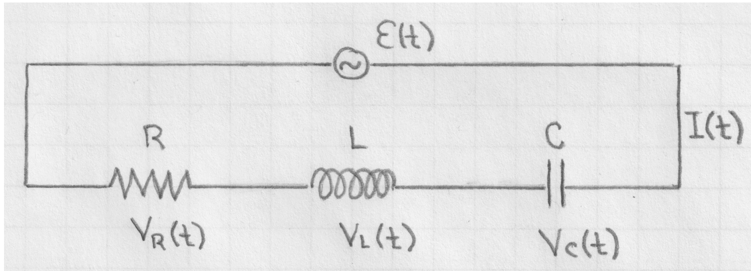
$$I(t) = I_R(t) = I_R(t) = I_C(t) = I_\mathcal{E} e^{\imath(\omega t - \delta_\mathcal{E})}. \tag{4}$$

The instantaneous voltages across the three devices add up to the instantaneous voltage (1) supplied by the ac source (loop rule):

$$\mathcal{E}(t) = V_R(t) + V_L(t) + V_C(t) = V_\mathcal{E} e^{\imath\omega t}. \tag{5}$$

The voltages across individual devices are all related to the same current as established earlier:

$$V_R(t) = X_R I(t), \quad V_L(t) = X_L I(t), \quad V_C(t) = X_C I(t). \tag{6}$$



Substitution of (4) and (6) with device reactances,

$$X_R = R, \quad X_L = \imath\omega L, \quad X_C = \frac{1}{\imath\omega C}, \tag{7}$$

as identified earlier for single-device circuits, into (5) yields the relation,

$$V_\mathcal{E} e^{\imath\omega t} = \left(R + \imath\omega L + \frac{1}{\imath\omega C}\right) I_\mathcal{E} e^{\imath(\omega t - \delta_\mathcal{E})}, \tag{8}$$

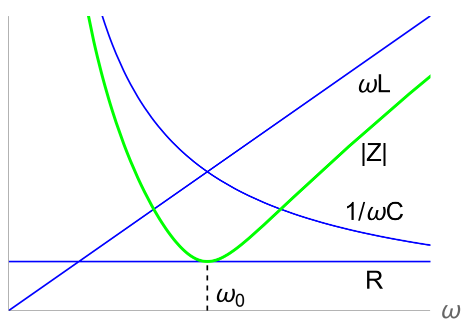
from which we extract the impedance (3) and the associated phase angle:

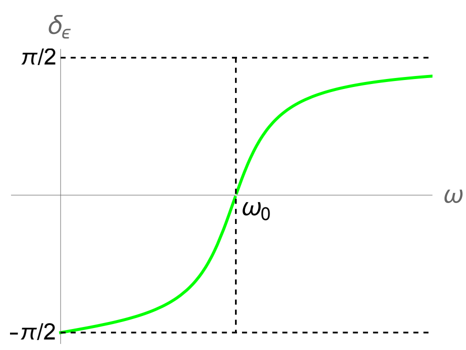
$$Z = R + \imath\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{\imath\delta_\mathcal{E}},$$

$$\delta_\mathcal{E} = \arctan\left(\frac{\omega L - 1/\omega C}{R}\right). \tag{9}$$

Graphical representations of both quantities, specifically their dependence on the driving frequency ω are shown below. Also shown are the magnitudes of the device reactances.

All plots and simplified expressions pertain to $L = 2$, $C = 1$, and $R = 0.5$ in SI units. Some adjustments are necessary if the order of size changes.





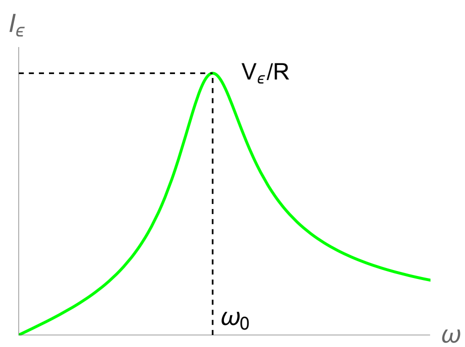
At low (high) frequency the capacitor (inductor) is the dominant device. At resonance, $\omega_0 = 1/\sqrt{LC}$, the impedance is purely resistive. The same shift of dominance is also reflected in the phase angle.

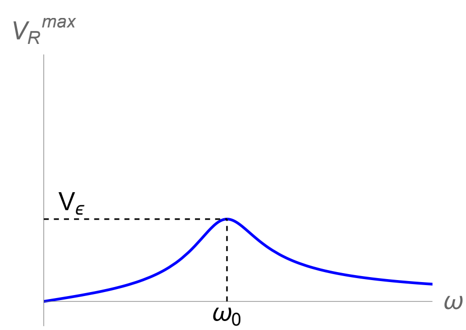
Resonance in the RLC series circuit is associated with a maximum in the current amplitude as shown below.

The voltages across the individual devices respond differently to variations in ω. Starting from (6) we can write

$$
\begin{aligned}
V_R(t) &= RI(t) = \frac{R}{|Z|} V_{\mathcal{E}} e^{\imath(\omega t - \delta_{\mathcal{E}})} = V_R^{max}(\omega) e^{\imath(\omega t - \delta_{\mathcal{E}})}, \\
V_L(t) &= \imath\omega L I(t) = \frac{\omega L}{|Z|} V_{\mathcal{E}} e^{\imath(\omega t - \delta_{\mathcal{E}} + \pi/2)} = V_L^{max}(\omega) e^{\imath(\omega t - \delta_{\mathcal{E}} + \pi/2)}, \\
V_C(t) &= \frac{I(t)}{\imath\omega C} = \frac{1}{\omega C|Z|} V_{\mathcal{E}} e^{\imath(\omega t - \delta_{\mathcal{E}} - \pi/2)} = V_C^{max}(\omega) e^{\imath(\omega t - \delta_{\mathcal{E}} - \pi/2)}.
\end{aligned}
\tag{10}
$$

The voltage amplitude across the resistor typically has a low maximum, $V_R^{max}(\omega_0) = V_{\mathcal{E}}$, at $\omega_0 = 1/\sqrt{LC}$.





The voltage amplitudes across the inductor and the capacitor typically have much higher maxima [lex193],

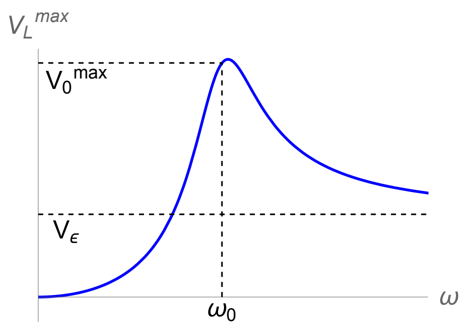
$$
V_L^{max}(\omega_L) = V_C^{max}(\omega_C) = \frac{V_0^{max}}{\sqrt{1 - \tau_{RC}/4\tau_{RL}}}, \quad V_0^{max} = V_{\mathcal{E}} \sqrt{\frac{\tau_{RL}}{\tau_{RC}}}, \tag{11}
$$

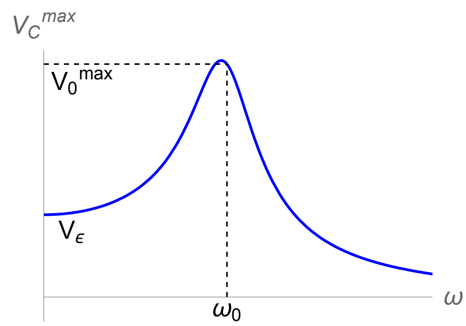
at the shifted frequencies,

$$
\omega_L = \frac{\omega_0}{\sqrt{1 - \tau_{RC}/2\tau_{RL}}} > \omega_0, \quad \omega_C = \omega_0 \sqrt{1 - \tau_{RC}/2\tau_{RL}} < \omega_0. \tag{12}
$$

In these expressions we have used the relaxation times associated with RC and RL circuits, which are in relation to the resonance frequency as follows:

$$
\tau_{RC} = RC, \quad \tau_{RL} = \frac{L}{R}, \quad \tau_{RC}\tau_{RL} = \frac{1}{\omega_0^2}. \tag{13}
$$





The large current amplitude near resonance is only indirectly caused by the voltage $V_\mathcal{E}$ supplied. That voltage amplified in the resonating circuit is what drives the current.

RLC parallel circuit:

Here the voltage (1) supplied by the ac source acts across each device:

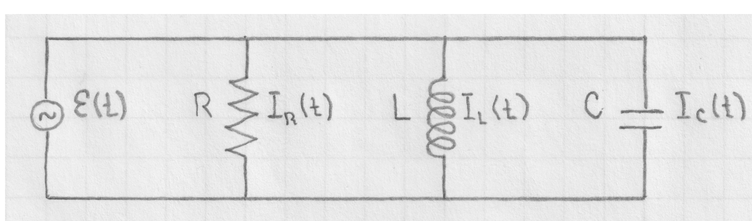
$$\mathcal{E}(t) = V_R(t) = V_L(t) = V_C(t) = V_\mathcal{E} e^{\imath\omega t}. \tag{14}$$

The instantaneous current through the three devices add up to the instantaneous current (2) that flows through the voltage source:

$$I(t) = I_R(t) + I_L(t) + I_C(t) = I_\mathcal{E} e^{\imath(\omega t - \delta_\mathcal{E})}. \tag{15}$$

The currents through individual devices are all related to the same voltage as established earlier:

$$I_R(t) = \frac{\mathcal{E}(t)}{X_R}, \quad I_L(t) = \frac{\mathcal{E}(t)}{X_L}, \quad I_C(t) = \frac{\mathcal{E}(t)}{X_C}. \tag{16}$$



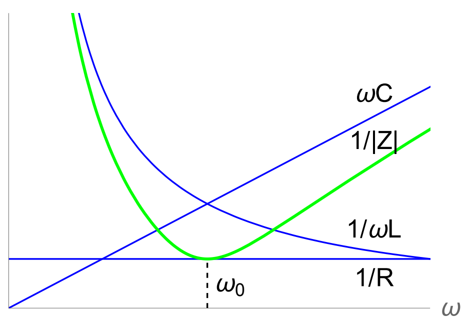
Substitution of (14) and (16) into (15) yields the relation,

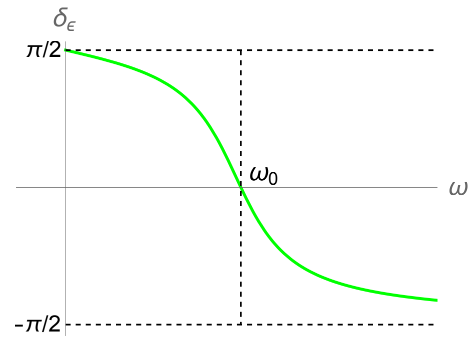
$$V_\mathcal{E} e^{\imath\omega t} \left(\frac{1}{R} + \frac{1}{\imath\omega L} + \imath\omega C \right) = I_\mathcal{E} e^{\imath(\omega t - \delta_\mathcal{E})}, \tag{17}$$

from which we extract the impedance (3) and the associated phase angle:

$$Z = \left[\frac{1}{R} + \imath\left(\omega C - \frac{1}{\omega L}\right)\right]^{-1} = \left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{-1/2} e^{\imath\delta_{\mathcal{E}}},$$
$$\delta_{\mathcal{E}} = \arctan\left(\frac{1/\omega L - \omega C}{1/R}\right). \tag{18}$$

Graphical representations of $1/|Z|$ and $\delta_{\mathcal{E}}$ versus ω are shown below. Also shown are the magnitudes of the inverse device reactances. All plots and simplified expressions pertain to $L = 2$, $C = 1$, and $R = 3$ in SI units.

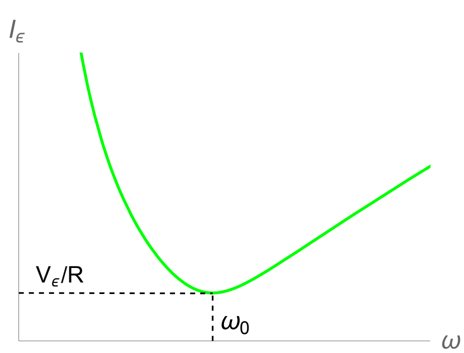


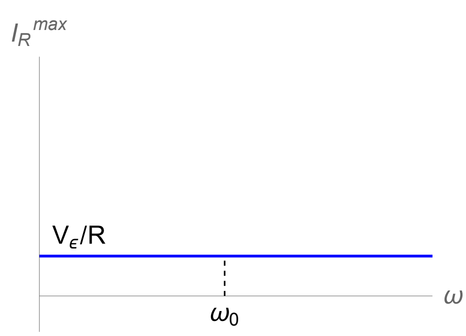


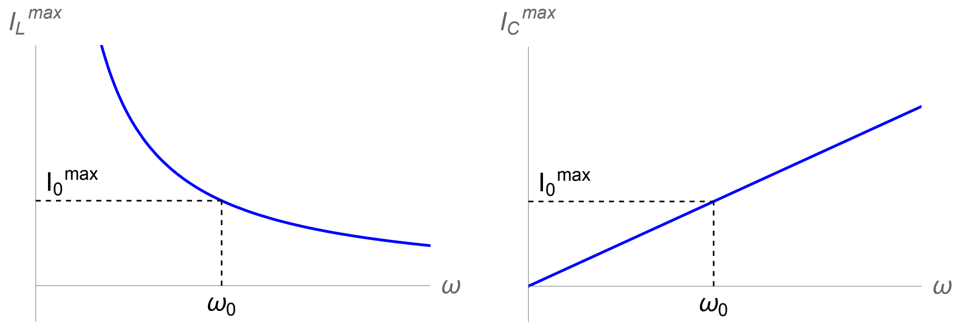
Here it's the currents through the individual devices that repsond differently to variations of ω. Starting from (16) we can write

$$I_R(t) = \frac{\mathcal{E}(t)}{R} = \frac{V_{\mathcal{E}}}{R} e^{\imath\omega t} = I_R^{max} e^{\imath\omega t},$$
$$I_L(t) = \frac{\mathcal{E}(t)}{\imath\omega L} = \frac{V_{\mathcal{E}}}{\omega L} e^{\imath(\omega t - \pi/2)} = I_L^{max}(\omega) e^{\imath(\omega t - \pi/2)},$$
$$I_C(t) = \imath\omega C\mathcal{E}(t) = \omega C V_{\mathcal{E}} e^{\imath(\omega t + \pi/2)} = I_C^{max}(\omega) e^{\imath(\omega t + \pi/2)}. \tag{19}$$

The amplitudes of the current through the power supply and the individual devices versus ω are shown separately below.







The currents through the inductor and capacitor have opposite phase for any value of ω. At resonance they have the same amplitude,

$$I_L^{max}(\omega_0) = I_C^{max}(\omega_0) = V_\mathcal{E}\sqrt{\frac{C}{L}}. \tag{20}$$

At low frequency the current through the inductor is largest and at high frequency the current through the capacitor.

The current through the power supply has the lowest amplitude at resonance even though large currents may flow through the reactive devices. At resonance the currents through the inductor and capacitor cancel each other at the junctions.

Circuits with other RLC combinations:

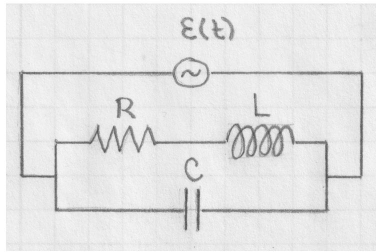
There are three combinations with two devices in series and three with two devices in parallel. We limit their analysis to the calculation of the impedance (3), specifically its magnitude $|Z|$ and its phase angle $\delta_\mathcal{E}$ as functions of ω.

The task is made simple by the use of the complex device reactances (7) in combination with Kirchhoff's laws [lln11].

Keep in mind that $|Z|$, R, ωL, $1/\omega C$ have units $[\Omega]$, $LC = 1/\omega_0^2$ is the square of an inverse (angular) frequency, and $RC = \tau_{RC}$, $L/R = \tau_{RL}$ are relaxation times (inverse frequencies).

In all plots of $|Z|$ and $\delta_\mathcal{E}$ we set $L = 1$ and $C = 1$ implying $\omega_0 = 1$, all in SI units. We set R in each case such as to highlight features of interest.

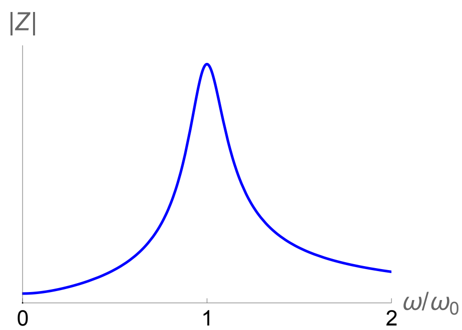
Case #1:

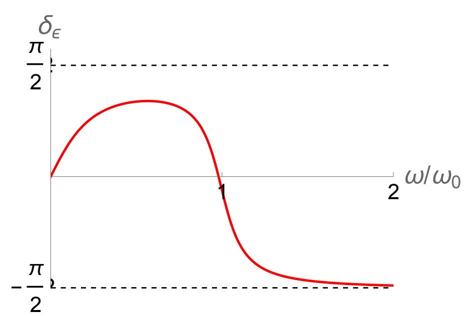


$$Z = \left(\frac{1}{R + \imath\omega L} + \imath\omega C \right)^{-1}, \tag{21}$$

$$|Z| = \sqrt{\frac{R^2 + (\omega L)^2}{(\omega C)^2[R^2 + (\omega L - 1/\omega C)^2]}}, \tag{22}$$

$$\delta_{\mathcal{E}} = \arctan\left(\frac{\omega L}{R} \left[1 - LC\omega^2 - \frac{CR}{L/R} \right] \right). \tag{23}$$



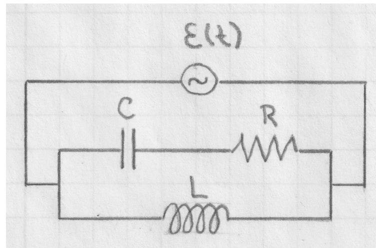


At low ω, the resistor is the dominant device. The capacitor blocks any significant current and the inductor is close to invisible.

With increasing ω, the RL current decreases and the C current increases. The phase of the RL current increases, in growing opposition to the phase of the C current. Z has a maximum at $\omega = \omega_0$. Here the resulting current has a minimum and $\delta_{\mathcal{E}} = 0$.

At high ω, the capacitor becomes the dominant device. The inductor increasingly blocks RL current. The rapidly alternating C current leaves the capacitor largely uncharged. Its phase angle is negative.

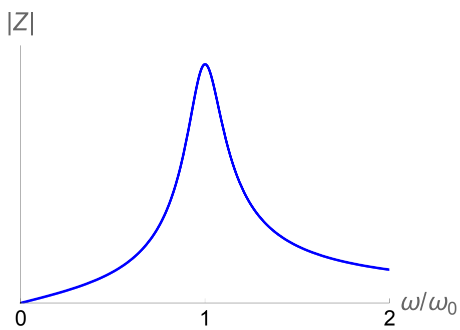
Case #2:

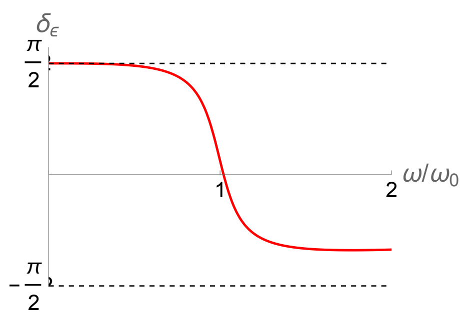


$$Z = \left(\frac{1}{1/\imath\omega C + R} + \frac{1}{\imath\omega L} \right)^{-1}, \tag{24}$$

$$|Z| = \omega L \sqrt{\frac{1 + (RC\omega)^2}{(RC\omega)^2 + (LC\omega^2 - 1)^2}}, \tag{25}$$

$$\delta_{\mathcal{E}} = \arctan\left(\frac{1 + LC\omega^2[RC/(L/R) - 1]}{(LC\omega^2)(RC\omega)} \right). \tag{26}$$





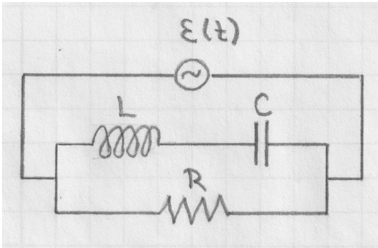
At low ω, almost all current is L current. The impedance is very low and the phase angle is strongly positive.

With ω increasing, the RC current increases and the L current decreases. The combined current decreases. $|Z|$ reaches a maximum at $\omega = \omega_0$, where phase angle changes sign.

At higher ω, the RC current begins to dominate as the L current is progressively suppressed. The phase angle goes negative and the overall current increases again as $|Z|$ decreases..

At very high ω (not shown), the current is controlled by the resistor alone. The inductor becomes a current stopper and the capacitor becomes transparent. Asymptotically for $\omega \to \infty$ we have $|Z| \to R$ and $\delta_{\mathrm{E}} \to 0$.

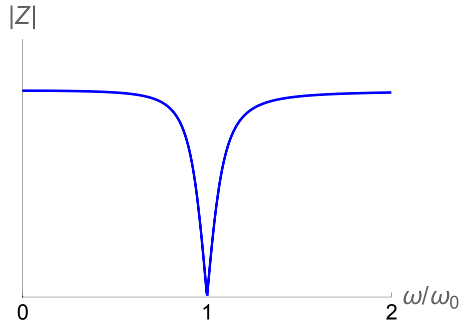
Case #3:

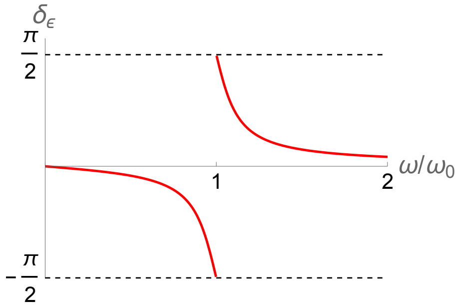


$$Z = \left(\frac{1}{1/\imath\omega C + \imath\omega L} + \frac{1}{R} \right)^{-1}, \tag{27}$$

$$|Z| = \frac{R|LC\omega^2 - 1|}{\sqrt{(RC\omega)^2 + (LC\omega^2 - 1)^2}}, \tag{28}$$

$$\delta_{\mathcal{E}} = \arctan\left(\frac{RC\omega}{LC\omega^2 - 1} \right). \tag{29}$$



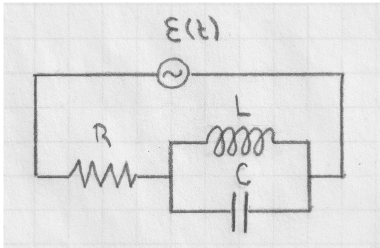


The R current has an ω-independent amplitude. The LC current is blocked by the capacitor in the low-ω limit and by the inductor in th high-ω limit.

At $\omega = \omega_0$ the LC current resonates. Its amplitude diverges. The impedance touches zero. The phase angle is discontinuous.

The ω-dependence of the phase angle echoes the dominance of the resistive device at low or high ω and the dominace of he reactive devices near the resonce frequency.

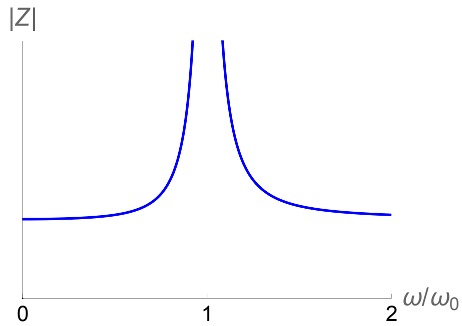
Case #4:

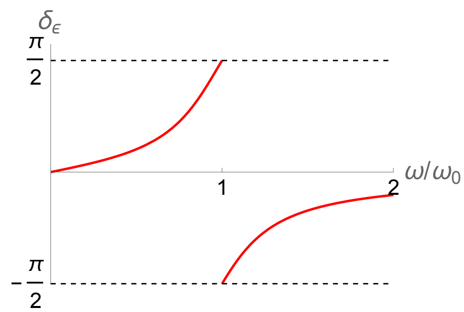


$$Z = R + \left(\frac{1}{\imath\omega L} + \imath\omega C\right)^{-1}, \tag{30}$$

$$|Z| = \sqrt{R^2 + \left(\frac{\omega L}{1 - LC\omega^2}\right)^2}, \tag{31}$$

$$\delta_{\mathcal{E}} = \arctan\left(\frac{\omega L}{R(1 - LC\omega^2)}\right). \tag{32}$$





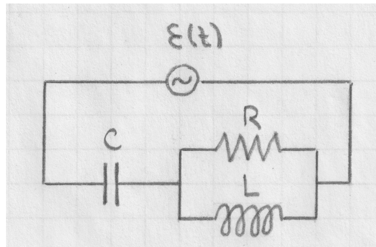
This case shares with case #3 an undamped resonance at $\omega = \omega_0$. Here the impedance diverges at resonance. The L and C currents cancel each other at the two junctions and thus block any current through the resistor.

In the low-ω limit, the capacitor blocks all current and the inductor becomes transparent, leaving the resistor in control of the current.

In the high-ω limit, it's the inductor that blocks all current and the capacitor that becomes invisible, again leaving the resistor in control of the current.

The shift of dominance between the resistive device and the reactive devices is again reflected in the ω-dependence of the phase angle.

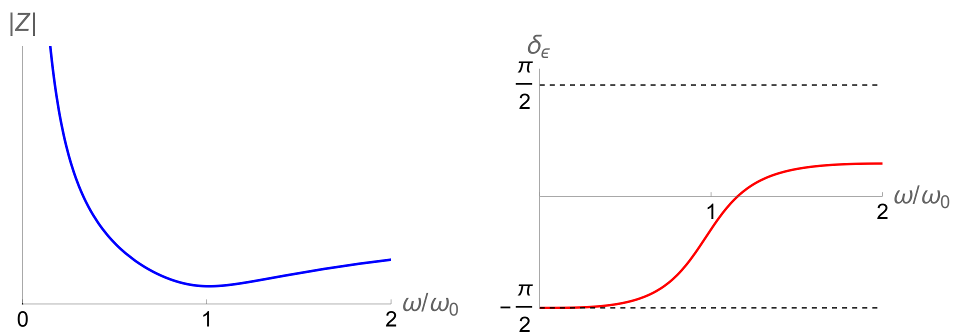
Case #5:



$$Z = \frac{1}{\imath\omega C} + \left(\frac{1}{R} + \frac{1}{\imath\omega L}\right)^{-1}, \tag{33}$$

$$|Z| = \sqrt{\frac{(\omega L)^2 + R^2(LC\omega^2 - 1)^2}{(LC\omega^2)^2 + (RC\omega)^2}}. \tag{34}$$

$$\delta_{\mathrm{E}} = \arctan\left(\frac{(RC\omega)^2[LC\omega^2 - 1] - (LC\omega^2)^2}{(RC\omega)(LC\omega^2)^2}\right). \tag{35}$$

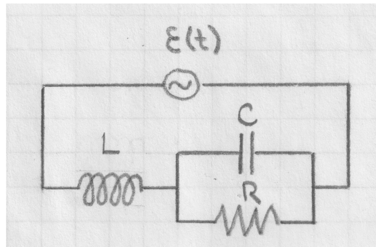


The capacitor blocks all current in the low-ω limit. Here the impedance diverges. The phase angle at $-\pi/2$ confirms the dominance of the capacitor.

In the high-ω limit, the capacitor becomes transparent and the inductor forces all current to go through the resistor. The impedance approaches R and the phase angle approaches zero (not shown).

Near $\omega = \omega_0$ the impedance has a minimum and the phase angle changes sign. Here the voltages across the capacitor and the inductor are close to opposed in phase, which facilitates a large current.

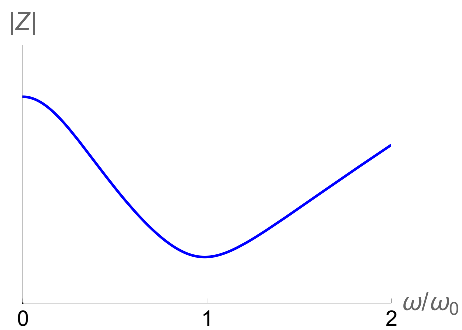
Case #6:

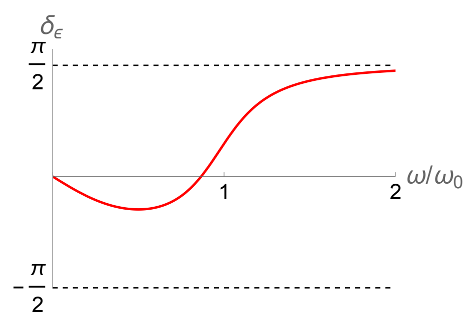


$$Z = \imath\omega L + \left(\imath\omega C + \frac{1}{R}\right)^{-1}, \tag{36}$$

$$|Z| = \sqrt{\frac{(\omega L)^2 + R^2(LC\omega^2 - 1)^2}{1 + (RC\omega)^2}}, \tag{37}$$

$$\delta_{\mathrm{E}} = \arctan\left(\frac{L}{R}\omega + RC\omega[LC\omega^2 - 1]\right). \tag{38}$$





In the low-ω limit, the capacitor forces all current to go through the resistor. The inductor has zero impedance in that limit. Hence the resistor is dominant, the impendance is nonzero and finite. The phase angle is zero.

At high ω the overall current is controlled (and suppressed) by the inductor. The phase angle approaches $\pi/2$ as the impedance diverges.

Near $\omega = \omega_0$ the impedance has a minimum and the phase angle changes sign as in case #5. The roles of inductor and capacitor interchanged. A large current is facilitated again by opposite voltages across the reactive devices.