Rapidities [lam26]

Here we revisit the longitudinal velocity addition rule.

If a particle moves with velocity u' in frame $\mathcal{F}'$, which, in turn, moves with velocity v relative to frame $\mathcal{F}$ (both in the same direction), then the velocity
[lln16] of the particle as seen from frame $\mathcal{F}$ is

$$u = \frac{u' + v}{1 + u'v/c^2}. \tag{1}$$

This is a consequence of the Lorentz invariance of relativistic kinematics. It ensures that the addition of any velocities $u', v < c$ yields a velocity $u < c$.

For $u', v \ll c$, the simpler addition rule,

$$u = u' + v, \tag{2}$$

reflecting the Galilean invariance of nonrelativistic kinematics.

In some contexts of relativistic kinematics, it is useful to operate with variables that satisfy addition rule (2) instead of (1) without violating Lorentz invariance. Such variables are named *rapidities*.

If we set

$$\frac{u'}{c} = \tanh\psi', \quad \frac{v}{c} = \tanh\phi, \quad \frac{u}{c} = \tanh\psi, \tag{3}$$

then Eq. (1) turns into

$$\tanh\psi = \frac{\tanh\psi' + \tanh\phi}{1 + \tanh\psi' \tanh\phi}. \tag{4}$$

At this point we recall one of the geometric identity for hyperbolic functions,

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}, \tag{5}$$

valid for variables with range $-\infty < x, y < \infty$. From a comparison of (4) and (5) we conclude that the rapidities defined in (3) indeed satisfy the simpler addition rule (2):

$$\psi = \psi' + \phi. \tag{6}$$

The range $-1 < \tanh x < 1$ of the hyperbolic tangent ensures that all velocites remain subliminary when rapidites of any values are added as in (6).

Rapidities as defined in (3) also assume the role of 'imaginary angles' when we interpret the Lorentz transformation as an orthogonal transformation in 4D spacetime.

The rotation of a Cartesian coordinate system about the z axis in 3D space,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{7}$$

is an orthogonal transformation. It leaves distances invariant. The equality,

$$\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + z^2}, \tag{8}$$

is readily verified. The inverse transformation matrix, obtained by reversing the rotation angle α, is its transpose (hallmark of orthogonality):

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The Lorentz transformation associated with a velocity boost in x direction,
[lln25] as introduced later, is a linear transformation in 4D spacetime:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \quad \beta \doteq \frac{v}{c}, \quad \gamma \doteq \frac{1}{\sqrt{1-\beta^2}}, \tag{10}$$

Expressing the elements of the transformation matrix in terms of the rapidity ϕ from before, using the relations,

$$\begin{aligned} \beta = \tanh\phi = \frac{\sinh\phi}{\cosh\phi} \quad &\Rightarrow\ 1 - \beta^2 = \frac{\cosh^2\phi - \sinh^2\phi}{\cosh^2\phi} = \frac{1}{\cosh^2\phi} \\ \cosh\phi = \frac{1}{\sqrt{1-\beta^2}} = \gamma, \quad &\sinh\phi = \tanh\phi\cosh\phi = \beta\gamma, \end{aligned} \tag{11}$$

we arrive at a relation akin to (9):

$$\begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

The (squared) invariant spacetime distances, which, unlike distances in 3D Euclidean space, can be negative, include

$$x'^2 + y'^2 + z'^2 - (ct')^2 = x^2 + y^2 + z^2 - (ct)^2. \tag{13}$$