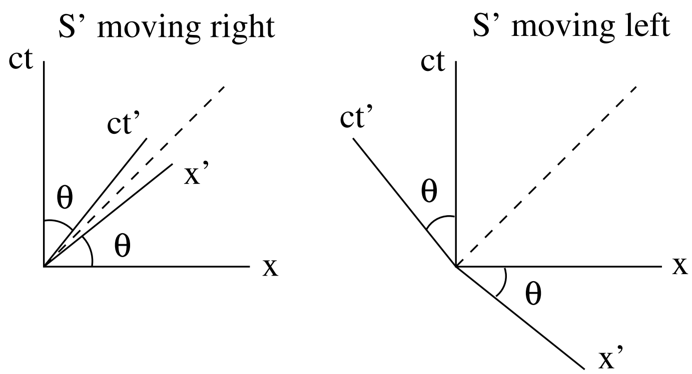
Twin Paradox [lam25]

Here we use Minkowski diagrams to illustrate the famous twin paradox.

Jack and Jill are twins. They synchronize their watches (and calendars) at $t = t' = 0$. Then Jill travels into space at $v = \frac{4}{5}c$ for some time, turns around, and returns at the same speed. Time dilation: $t'/t = \frac{3}{5}$.

Turning around is, of course, a matter in need of explanation (supplied later).

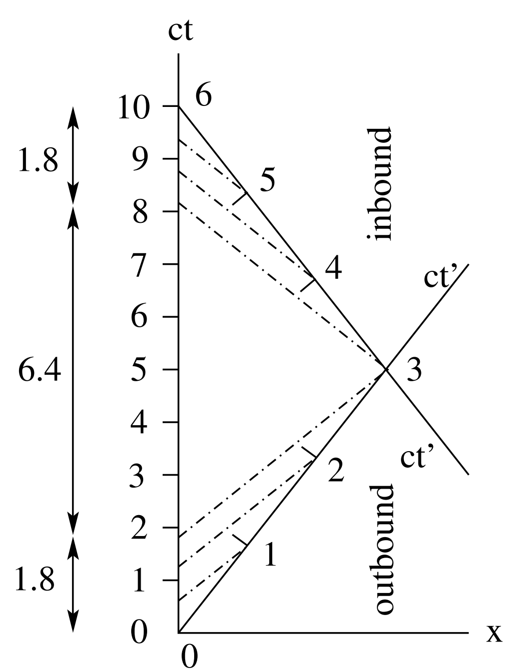


Jack's perspective:

- While Jill is outbound
  she ages 3 years
  and he ages 5 years.
- Jill turns around quickly.
- While Jill is inbound
  she ages 3 years
  and he ages 5 years.

Jill's perspective:

- While outbound
  she ages 3 years
  and Jack ages 1.8 years.
- As she turns around quickly
  Jack ages 6.4 years.
- While inbound
  she ages 3 years
  and Jack ages 1.8 years.



[lex144]

A person on a round trip necessarily involves accelerated motion. Jill might loop around a massive object for that purpose.

Clocks low in gravitational potential (close to a massive object) go more slowly than clocks higher in potential (further away from massive objects), says general relativity. Time dilation: $t_{\rm high}/t_{\rm low} - 1 \simeq gh/c^2$.

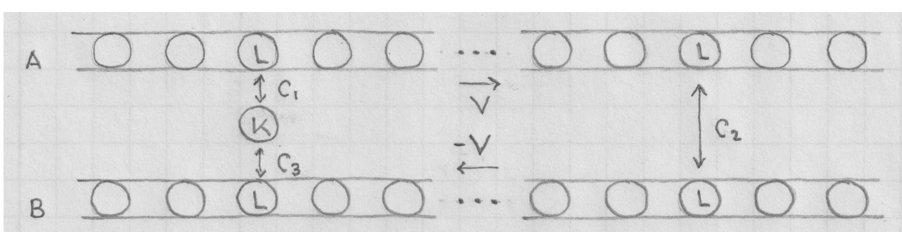
However, the twin paradox can be fully understood without resorting to accelerated motion.

Consider Jack (K) standing at a station between two long trains A and B moving at constant velocity $v = \frac{4}{5}c$ relative to Jack in opposite directions.

Each train has synchronized clocks in equidistant windows and a passenger looking out.

Consider three events (and replace years by hours).

1. Jack compares the station clock with the clock on train A that happens to be opposite himself. The passenger at that window is Jill (L).
2. After $\Delta t'_{12} = 3$h (hours), Jill compares the time of her clock with the time on the clock in train B at the window opposite her, where another Jill sits.
3. After an equal time interval, $\Delta t'_{23} = 3$h, Jill in train B arrives at Jack's station and compares the time of her clock with that of the station clock. Jack's clock has advanced $\Delta t_{13} = 10$h. The relayed timepiece of the two Jills has advanced only $\Delta t'_{13} = 6$h.



Quantitative analysis:

- Time dilation due to relative motion between station (unprimed) and each train (primed) depending on perspective:

$$\frac{\Delta t}{\Delta \tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{5}{3} \quad \text{or} \quad \frac{\Delta t'}{\Delta \tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{5}{3}.$$

- Total travel time between events 1 and 3 as recorded by clocks onboard:

$$\Delta t'_{13} = \Delta t'_{12} + \Delta t'_{23} = 3\text{h} + 3\text{h} = 6\text{h}.$$

- Each train travels the proper distance ℓ_0 between successive events (measured in light hours):

$$\ell_0 = \Delta t'_{12} v = \Delta t'_{23} v = 3 \cdot \frac{4}{5}\,\text{lh} = \frac{12}{5}\,\text{lh}.$$

- The station clock as seen from each train has dilated time intervals. The sum of the two travel times as recorded by the station clock is

$$\Delta t_{12} + \Delta t_{23} = \frac{3}{5}(3\text{h} + 3\text{h}) = \frac{18}{5}\,\text{h}.$$

- If the train clocks in event 2 are synchronized at the position of the
[lln16] two Jills, then they are out of sync at the station:

$$\Delta t_{\text{as}} = 2\,\frac{\ell_0 v/c^2}{\sqrt{1 - v^2/c^2}} = 2\left(\frac{12}{5}\right)\left(\frac{4}{5}\right)\left(\frac{5}{3}\right)\,\text{h} = \frac{32}{5}\,\text{h}.$$

  Total time between events 1 and 3 as recorded by station clock:

$$\Delta t_{13} = \Delta t_{12} + \Delta t_{23} + \Delta t_{\text{as}} = \frac{18}{5}\,\text{h} + \frac{32}{5}\,\text{h} = 10\text{h}.$$