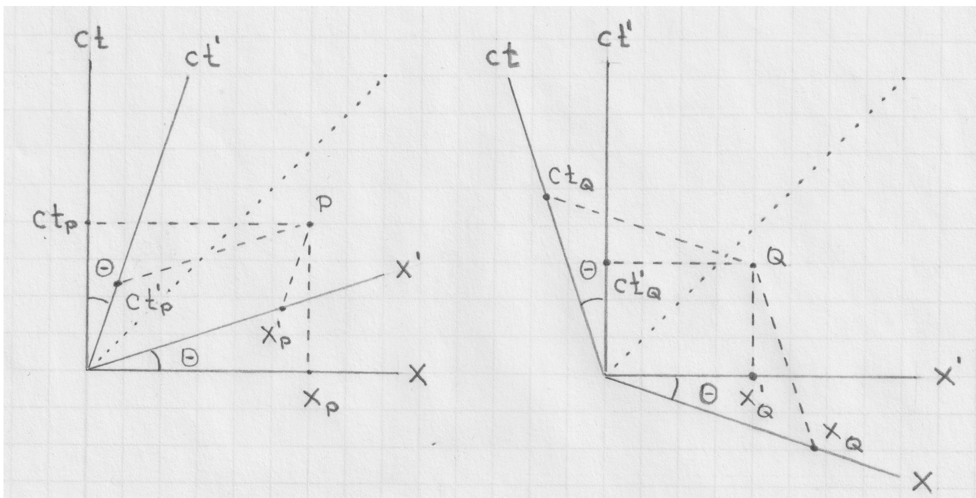
Minkowski Diagram [lam24]

The relativistic features described in the previous sections can be illustrated geometrically in a Minkowski diagram.

Consider two inertial frames: $\mathcal{F}'$ moving relative to $\mathcal{F}$ with velocity v in the positive x-direction. Clocks are synchronized at $t = t' = 0$ and $x = x' = 0$.

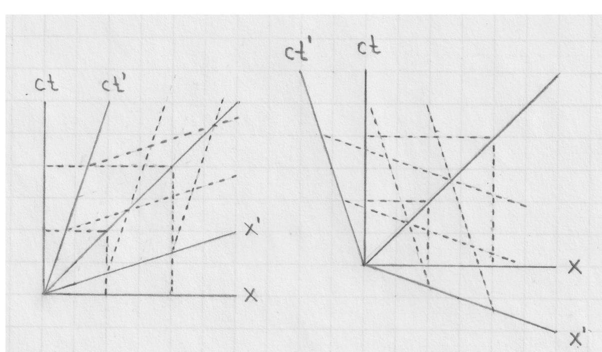
The Minkowski diagram has two sets of axes: a rectangular set for one frame and a set with tilted angles for the other. The tilt angle is $\theta = \pm \arctan(v/c)$.



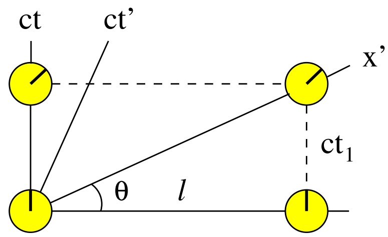
The coordinates of an event P or Q in either frame are determined by projections as shown.

The axes of the two frames in each diagram have different scales (shown later). The (dotted) world line of light is the same in both frames: $x = ct$ and $x' = ct'$.

The equality of the speed of light in the two frames is the unit slope of the world line of light in both frames as illustrated in the two Minkowski diagrams below.



The relativity of simultaneity can be neatly illustrated by pairs of synchronized clocks in one frame viewed from a frame in relative motion.



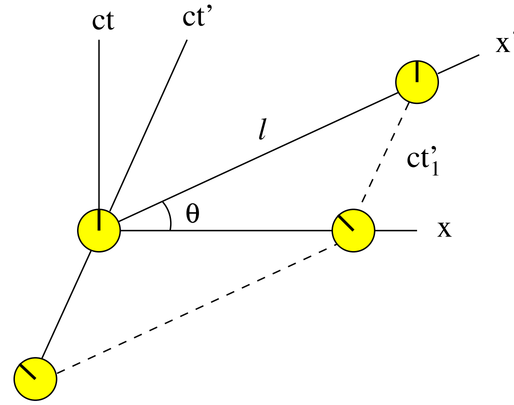


Diagram on the left:

- Two synchronized clocks at rest in frame $\mathcal{F}$ shown at $t = 0$ and at $t = t_1 = \ell v/c^2$.
- Observers in frame $\mathcal{F}'$ see the two clock moving left.
- Clock readings in frame $\mathcal{F}'$ at $t' = 0$ are taken on the x'-axis, showing the clock behind in space ahead in time.

Diagram on the right:

- Two synchronized clocks at rest in frame $\mathcal{F}'$ shown at $t' = 0$ and at $t' = t'_1 = -\ell v/c^2$.
- Observers in frame $\mathcal{F}'$ see the two clock moving right.
- Clock readings in frame $\mathcal{F}$ at $t = 0$ are taken on the x-axis, showing again that the clock which is spatially behind is temporally ahead.

Minkowski diagrams do not preserve angles and scales. The units on the primed and unprimed axes are related by the following scale factor:

$$\frac{S'}{S} = \sqrt{\left(1 + \frac{v^2}{c^2}\right) \Big/ \left(1 - \frac{v^2}{c^2}\right)}.$$

Consider a right triangle with unit horizontal side and vertical side $v/c = \tan\theta$. The hypotenuse with length $\sqrt{1 + v^2/c^2}$ must then represent the contracted length $\sqrt{1 - v^2/c^2}$ of a unit length in $\mathcal{F}$ as measured in $\mathcal{F}'$.

This requirement can be accommodated by stretching the length scale in $\mathcal{F}'$ by a factor S'/S. An analogous argument pertains to the time scale.

In the illustration below we use

$$\frac{v}{c} = 0.6 \quad \Rightarrow \sqrt{1 - \frac{v^2}{c^2}} = 0.8 \quad \Rightarrow \frac{S'}{S} \simeq 1.46.$$

If the unit of time is [min] then the unit of length is [lmin].

Length contraction:

Line (i): Rod of proper length $\ell = 5$ at rest in $\mathcal{F}$ viewed from $\mathcal{F}'$:

$$\ell' = \ell\sqrt{1 - v^2/c^2} = 4.$$

Line (ii): Rod of proper length $\ell' = 5$ at rest in $\mathcal{F}'$ viewed from $\mathcal{F}$:

$$\ell = \ell'\sqrt{1 - v^2/c^2} = 4.$$

Time dilation:

Consider arrays of clocks in $\mathcal{F}$ and $\mathcal{F}'$ synchronized at $t = t' = 0$.

Line (iii): Moving clock in $\mathcal{F}'$ (showing time $ct' = 4$) viewed in $\mathcal{F}$ at position $x = 0$ and time $ct = 5$:

$$ct = \frac{ct'}{\sqrt{1 - v^2/c^2}}.$$

Line (iv): Moving clock in $\mathcal{F}$ (showing time $ct = 4$) viewed in $\mathcal{F}'$ at position $x' = 0$ and time $ct' = 5$:

$$ct' = \frac{ct}{\sqrt{1 - v^2/c^2}}.$$

lex163][lex164]
[lex166]