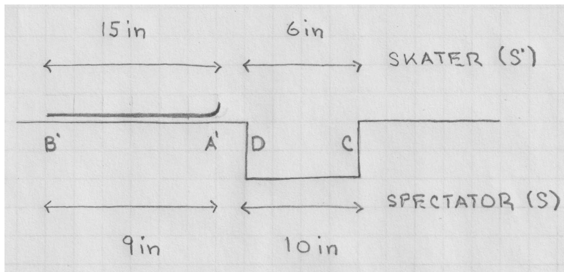
Skater Paradox [lam23]

A skater with blades of proper length $\ell_0 = 15\text{in}$ on his skates moves with velocity $v = 0.8c$ relative to a flat ice surface, approaching a gap in the ice of proper width $d_0 = 10\text{in}$. A spectator at rest on the surface watches the skater's encounter with the gap.



Skater's perspective (frame S'):

The gap in the ice is Lorentz contracted to a width $d = d_0\sqrt{1-(0.8)^2} = 6\text{in}$, which is shorter than the proper length $\ell_0 = 15\text{in}$ of his blades. The front end A' of the blade will gain support on the far side C of the gap before the rear end B' loses support on the near side D. The skater concludes that he will make it across the gap without accident.

Spectator's perspective (frame S):

The blades are Lorentz contracted to length $\ell = \ell_0\sqrt{1-(0.8)^2} = 9\text{in}$, shorter than the proper length $d_0 = 10\text{in}$ of the gap in the ice. The rear end B' of the blade loses support on the near side D of the gap before the front end A' is able to gain support on the far side C. The spectator concludes that the skater will not make it across the gap without accident.

Analysis:

Event 1: Rear end B' of blade enters gap at D.
Event 2: Front end A' of blade exits gap at C.

Frame S': $\Delta x' \doteq x'_2 - x'_1 = \ell_0 = 15\text{in}$,
$\Delta t' \doteq t'_2 - t'_1 = -(15\text{in} - 6\text{in})/v = -9\text{in}/v$.
The result $\Delta t' < 0$ suggests a safe passage across the gap.

Frame S: $\Delta x \doteq x_2 - x_1 = d_0 = 10\text{in}$,
$\Delta t \doteq t_2 - t_1 = (10\text{in} - 9\text{in})/v = 1\text{in}/v$.
The result $\Delta t > 0$ suggests that an accident will happen.

The contradictory suggestions come from relative (frame-dependent) quantities. The true answer comes from a quantitiy, the Lorentz invariant distance in spacetime,

$$(\Delta s)^2 = (\Delta x)^2 - (c\Delta t)^2,$$

which is absolute (frame-independent).

$$\text{Frame } S': \ (\Delta s)^2 = (15\text{in})^2 - (-9\text{in}/0.8)^2 = (98.4375\text{in})^2 > 0.$$
$$\text{Frame } S: \ (\Delta s)^2 = (10\text{in})^2 - (1\text{in}/0.8)^2 = (98.4375\text{in})^2 > 0.$$

Events 1 and 2 have a space-like relationship. They have no definite time-ordering as demonstrated. Such events cannot be causally related.

Conclusion:

The skater implies a causal relation between the two events: the front end A' finds contact with ice before the rear end B' loses contact. This implication is fallacious at for two events in a space-like relationship. The spectator's opposite implication is equally fallacious for the same reason.

At lower speed (e.g. $v = 0.3c$), the two events have a time-like relationship: $(\Delta s)^2 < 0$, implying that the two events are consistently time-ordered in all frames. Event 2 happens before event 1. The skater and the spectator agree that the passage is safe.

The question regarding safety can only be answered for events with a time-like relationship.

Modified scenario:

Consider the case with blades of proper length $\ell_0 = 10\text{in}$ and a gap of proper length $d_0 = 15\text{in}$. In this case, trouble is obvious to skater and spectator at low speed. However, at high speed ($v = 0.8c$), the skater again feels safe.

The spacetime distance between the two events is space-like at high speed and time-like at low speed as in the original scenario.