# First- and Second-Order Correlation Functions [lam2]

Correlation functions are an instrument for the classification of photon streams which is complementary to the photon statistics discussed in [lln24].

In the following we aim to clarify the distinction between first-order and second-order correlation functions as employed in quantum optics.

**First-order correlation function:**

Consider a linearly polarized electromagnetic field, which is in state $|\psi\rangle$ prior to a photon detection and in state $|\psi'\rangle$ after the detection.

The transition probability,

$$|\langle\psi'|E^{(+)}(\mathbf{r},t)|\psi\rangle|^2 = \langle\psi|E^{(-)}(\mathbf{r},t)|\psi'\rangle\langle\psi'|E^{(+)}(\mathbf{r},t)|\psi\rangle,$$

which involves the mutually adjoint electric-field operators from [lam3], represents a process that annihilates single photons between states $|\psi\rangle$ and $|\psi'\rangle$.

The probability of transitions from initial state $|\psi\rangle$ to any final state becomes

$$w_1(\mathbf{r},t) \doteq \sum_{\psi'}\langle\psi|E^{(-)}(\mathbf{r},t)|\psi'\rangle\langle\psi'|E^{(+)}(\mathbf{r},t)|\psi\rangle = \langle\psi|E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)|\psi\rangle.$$

Completeness relation: $\sum_{\psi'}|\psi'\rangle\langle\psi'| = \mathcal{I}$.

Density operator of mixed state [lam1]: $\rho = \sum_{\psi} P_\psi|\psi\rangle\langle\psi|$.

Detection probability more generally: $w_1(\mathbf{r},t) = \mathrm{Tr}[\rho E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)]$.

First-order correlation function:

$$G^{(1)}(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) \doteq \mathrm{Tr}[\rho E^{(-)}(\mathbf{r}_1,t_1)E^{(+)}(\mathbf{r}_2,t_2)].$$

Relation between detection probability of single photon and first-order correlation function:

$$w_1(\mathbf{r},t) = G^{(1)}(\mathbf{r},\mathbf{r};t,t) \doteq \bar{G}^{(1)}(\mathbf{r},t).$$

Related quantities of importance are the two-time correlation function,

$$\hat{G}^{(1)}(\mathbf{r};t_1,t_2) \doteq G^{(1)}(\mathbf{r},\mathbf{r};t_1,t_2) = \langle E^{(-)}(\mathbf{r},t_1)E^{(+)}(\mathbf{r},t_2)\rangle,$$

and the normalized version of the general first-order correlation function,

$$g^{(1)}(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) \doteq \frac{G^{(1)}(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2)}{[\bar{G}^{(1)}(\mathbf{r}_1,t_1)\bar{G}^{(1)}(\mathbf{r}_2,t_2)]^{1/2}}.$$

**Second-order correlation function:**

Joint probability for the successive detection of two photons if the initial state is a pure quantum state:

$$\begin{aligned} w_2(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) &= \sum_{\psi'} |\langle\psi'|E^{(+)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_1, t_1)|\psi\rangle|^2 \\ &= \langle\psi|E^{(-)}(\mathbf{r}_1, t_1)E^{(-)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_1, t_1)|\psi\rangle. \end{aligned}$$

Generalization to the case of a mixed initial state:

$$w_2(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \mathrm{Tr}\big[\rho E^{(-)}(\mathbf{r}_1, t_1)E^{(-)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_1, t_1)\big].$$

Second-order correlation function:

$$\begin{aligned} &G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4; t_1, t_2, t_3, t_4) \\ &\qquad \doteq \mathrm{Tr}[\rho E^{(-)}(\mathbf{r}_1, t_1)E^{(-)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_3, t_3)E^{(+)}(\mathbf{r}_4, t_4)]. \end{aligned}$$

Relation between two-photon detection probability and second-order correlation function:

$$w_2(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2, \mathbf{r}_1; t_1, t_2, t_2, t_1) = \bar{G}^{(2)}(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2).$$

Normalized version of the general second-order correlation function:

$$\begin{aligned} &g^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4; t_1, t_2, t_3, t_4) \\ &\qquad \doteq \frac{G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4; t_1, t_2, t_3, t_4)}{[\bar{G}^{(1)}(\mathbf{r}_1, t_1)\bar{G}^{(1)}(\mathbf{r}_2, t_2)\bar{G}^{(1)}(\mathbf{r}_3, t_3)\bar{G}^{(1)}(\mathbf{r}_4, t_4)]^{1/2}}. \end{aligned}$$

The intensity correlation function is a special case where photons are detected at the same location ($\mathbf{r} = 0$) at times $t$ and $t + \tau$.

$$g^{(2)}(t, t+\tau) = \frac{\bar{G}^{(2)}(t, t+\tau)}{\bar{G}^{(1)}(t)\bar{G}^{(1)}(t+\tau)} = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle\langle I(t+\tau)\rangle}.$$

Under stationary conditions the $t$-dependence disappears. Intensity correlations then only depend on the time delay between successive detections:

$$\bar{g}^{(2)}(\tau) = \frac{\langle I(0)I(\tau)\rangle}{\langle I(0)\rangle^2}.$$

The brackets include a time average under these circumstances.