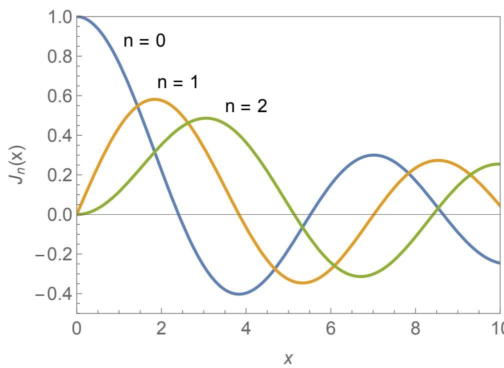
Bessel Functions [lam18]

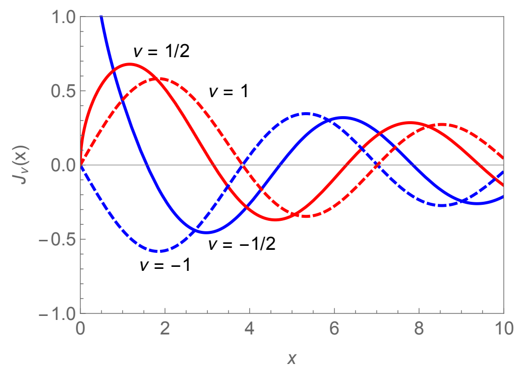
Solutions of partial differential equations for problems with cylindrical symmetry are often expressible as Bessel functions.

Bessel equation: $x^2R''(x) + xR'(x) + (x^2 - \nu^2)R(x) = 0$.

Bessel functions of the first kind: $\mathrm{J}_\nu(x) = \displaystyle\sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{2s+\nu}$. For noninteger ν, the function $\mathrm{J}_\nu(x)$ and $\mathrm{J}_{-\nu}(x)$ are linearly independent. Their linear dependence for integer ν is manifest in the relation,[1]

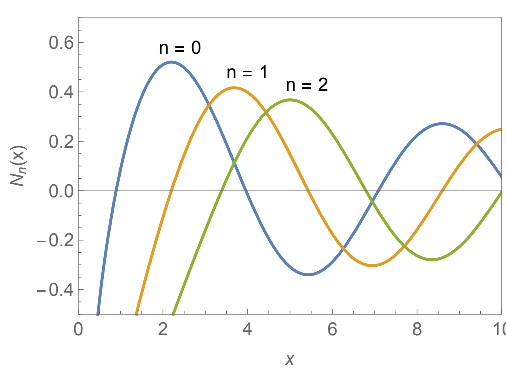
$$\mathrm{J}_{-\nu}(x) = (-1)^\nu \mathrm{J}_\nu(x) \quad : \ \nu \in \mathbb{Z}.$$





A solution which remains linearly independent of $\mathrm{J}_\nu(x)$ is the Neumann function (Bessel function of the second kind) constructed as follows:

$$\mathrm{N}_\nu(x) \doteq \frac{\mathrm{J}_\nu(x)\cos(\nu\pi) - \mathrm{J}_{-\nu}(x)}{\sin(\nu\pi)}.$$



The divergence at $x = 0$ is logarithmic in nature.

[1] This relation emerges on account of the fact that the function $\Gamma(x)$ diverges for non-positive integers.

Modified Bessel functions:

A change of sign in the parameter ν^2 of the Bessel equation produces solutions with quite different properties.

Modified Bessel equation: $x^2R''(x) + xR'(x) - (x^2 + \nu^2)R(x) = 0.$

Modified Bessel function of the first kind: $\mathrm{I}_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{2s+\nu}.$

For noninteger ν, the functions $\mathrm{I}_\nu(x)$ and $\mathrm{I}_{-\nu}(x)$ are again linearly independent, whereas for integer ν we have,

$$\mathrm{I}_{-\nu}(x) = \mathrm{I}_\nu(x) \quad : \ \nu \in \mathbb{Z}.$$

A solution which remains linearly independent of $\mathrm{I}_\nu(x)$ for integer ν is the MacDonald function (modified Bessel function of the second kind:

$$\mathrm{K}_\nu(x) \doteq \frac{\pi}{2}\frac{\mathrm{I}_{-\nu}(x) - \mathrm{J}_\nu(x)}{\sin(\nu\pi)}.$$