Meaning of Orthogonality [lam15]

Quantities of interest in electromagnetism are sometimes expressed as expansions in terms of orthogonal functions, just as vectors are expressed as components in a specific coordinate system.

3D space has three mutually perpendicular (orthogonal) directions. Orthogonal functions span a space with infinitely many mutually orthogonal directions in a generalized sense.

Consider the vector $\mathbf{a} = a_x\,\hat{\mathbf{i}} + a_y\,\hat{\mathbf{j}} + a_z\,\hat{\mathbf{k}}$. The three directions are encoded in the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, and their mutual orthogonality by the dot products $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$. The normalization of the unit vectors is also encoded in dot products: $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$.

The components of the vector $\mathbf{a}$ can be determined by projecting it onto the axes of the three mutually orthogonal directions. This geometric operation is encoded algebraically in the dot products, $a_x = \mathbf{a} \cdot \hat{\mathbf{i}}$, $a_y = \mathbf{a} \cdot \hat{\mathbf{j}}$, $a_z = \mathbf{a} \cdot \hat{\mathbf{k}}$.

Now consider a function $f(x)$ expressed as a linear combination (orthogonal expansion) of a set of orthogonal functions $\psi_m(x)$, which are mutually orthogonal and normalized in a generalized sense.

- ▷ Set of (real) orthogonal functions: $\psi_m(x), \quad m = 1, 2, \ldots$
- ▷ Range of variable: $x_{\text{min}} \leq x \leq x_{\text{max}}$.
- ▷ Weight function (specific to the set): $w(x)$.
- ▷ Mutual orthogonality: $\displaystyle\int_{x_{\text{min}}}^{x_{\text{max}}} dx\, w(x)\psi_m(x)\psi_{m'}(x) = 0 \quad : \; m \neq m'$.
- ▷ Normalization: $\displaystyle\int_{x_{\text{min}}}^{x_{\text{max}}} dx\, w(x)\psi_m(x)\psi_m(x) = 1$.
- ▷ Orthogonal expansion: $\displaystyle f(x) = \sum_n a_n \psi_n(x)$.
- ▷ Expansion coefficient: $\displaystyle a_m = \int_{x_{\text{min}}}^{x_{\text{max}}} dx\, w(x)\psi_m(x) f(x)$.

The last item is obtained by multiplying both sides of the previous item with $w(x)\psi_m(x)$ followed by integration.

$$\int_{x_{\text{min}}}^{x_{\text{max}}} dx\, w(x)\psi_m(x) = \sum_n a_n \int_{x_{\text{min}}}^{x_{\text{max}}} dx\, w(x)\psi_m(x)\psi_n(x) = \sum_n a_n \delta_{nm} = a_m.$$

This step is akin to the projection of a vector onto a coordinate axis as described above.

This scheme is readily to generalized to complex functions as are often employed in orthogonal expansions.

Note of caution: Convergence of such expansions is not, in general, guaranteed. Convergence theorems do exist.