Laplace Equation in Cylindrical Coordinates [lam14]

In generalization to the analysis presented in [lln7], we consider here a case which does not assume translational symmetry along the cylindrical axis.

Laplace equation: $\nabla^2\Phi(r,\phi,z) = 0$.

$$\Rightarrow \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

Product ansatz: $\Phi(r,\phi,z) = R(r)Q(\phi)Z(z)$.

Substitution: $QZ\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \frac{RZ}{r^2}\frac{d^2Q}{d\phi^2} + RQ\frac{d^2Z}{dz^2} = 0.$

Isolate variable ϕ: $\frac{r^2}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \frac{r^2}{Z}\frac{d^2Z}{dz^2} + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0.$

Separation constant for $Q(\phi)$: $-\nu^2$.

Solution: $\frac{d^2Q}{d\phi^2} = -\nu^2 Q \quad \Rightarrow \; Q(\phi) = ae^{\imath\nu\phi} + be^{-\imath\nu\phi}.$

For azimuthally periodic solutions we have $\nu \in \mathbb{Z}$.

Isolate variable z: $\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] - \frac{\nu^2}{r^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$

Separation constant for $Z(z)$: k^2 or $-\kappa^2$ with $k,\kappa \in \mathbb{R}$.

Solution #1: $\frac{d^2Z}{dz^2} = k^2 Z \quad \Rightarrow \; Z(z) = ce^{kz} + de^{-kz}.$

Solution #2: $\frac{d^2Z}{dz^2} = -\kappa^2 Z \quad \Rightarrow \; Z(z) = ce^{\imath\kappa z} + de^{-\imath\kappa z}.$

Boundary conditions associated with variable z determine which solution is physically relevant.

ODEs for variable r for case #1 (left) and case #2 (right):

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] + \left(k^2 - \frac{\nu^2}{r^2}\right) = 0, \quad \frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] - \left(\kappa^2 + \frac{\nu^2}{r^2}\right) = 0.$$

Bessel equation (left) for $x = kr$ and modified Bessel eq. (right) for $x = \kappa r$:

$$x^2R''(x) + xR'(x) + (x^2 - \nu^2)R(x) = 0, \quad x^2R''(x) + xR'(x) - (x^2 + \nu^2)R(x) = 0.$$

Solutions of the Bessel equation can be constructed from the following power-series with coefficients a_s determined recursively:

$$R_\pm(x) = x^{\pm\nu}\sum_{s=0}^{\infty} a_s x^s,$$

This leads to Bessel functions of the first kind [lln8]:

$$\mathrm{J}_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{2s+\nu}.$$

For noninteger ν, the functions $\mathrm{J}_\nu(x)$ and $\mathrm{J}_{-\nu}(x)$ are linearly independent. Their linear dependence for integer ν is manifest in the relation,

$$\mathrm{J}_{-\nu}(x) = (-1)^\nu \mathrm{J}_\nu(x) \quad : \; \nu \in \mathbb{Z}.$$

A solution which remains linearly independent of $\mathrm{J}_\nu(x)$ is the Neumann function (Bessel function of the second kind) constructed as follows:

$$\mathrm{N}_\nu(x) \doteq \frac{\mathrm{J}_\nu(x)\cos(\nu\pi) - \mathrm{J}_{-\nu}(x)}{\sin(\nu\pi)}.$$

Solutions of the modified Bessel equation can be constructed analogously in the form of modifed Bessel functions [lln8],

$$\mathrm{I}_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!\Gamma(s+\nu+1)} \left(\frac{x}{2}\right)^{2s+\nu}.$$

For noninteger ν, the functions $\mathrm{I}_\nu(x)$ and $\mathrm{I}_{-\nu}(x)$ are again linearly independent, whereas for integer ν we have,

$$\mathrm{I}_{-\nu}(x) = \mathrm{I}_\nu(x) \quad : \; \nu \in \mathbb{Z}.$$

A solution which remains linearly independent of $\mathrm{I}_\nu(x)$ is

$$\mathrm{K}_\nu(x) \doteq \frac{\pi}{2} \frac{\mathrm{I}_{-\nu}(x) - \mathrm{J}_\nu(x)}{\sin(\nu\pi)}.$$