Laplace Equation in Spherical Coordinates [lam13]

In generalization to the analysis presented in [lln7], we consider here a case which does not assume azimuthal symmetry.

Laplace equation: $\nabla^2\Phi(r,\theta,\phi)=0$.

$$\Rightarrow\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0.$$

Product ansatz: $\Phi(r,\theta,\phi)=\dfrac{U(r)}{r}\,P(\theta)Q(\phi)$.

Substitution: $PQ\,\dfrac{d^2U}{dr^2}+\dfrac{UQ}{r^2\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\,\dfrac{dP}{d\theta}\right)+\dfrac{UP}{r^2\sin^2\theta}\,\dfrac{d^2Q}{d\phi^2}=0.$

Isolate variable ϕ: $r^2\sin^2\theta\left[\dfrac{1}{U}\dfrac{d^2U}{dr^2}+\dfrac{1}{Pr^2\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\,\dfrac{dP}{d\theta}\right)\right]+\dfrac{1}{Q}\dfrac{d^2Q}{d\phi^2}=0.$

Judicious choice of separation constant: $m^2,\quad m\in\mathbb{Z}$.

Periodic solutions for $Q(\phi)$: $\dfrac{d^2Q}{d\phi^2}=-m^2Q\quad\Rightarrow\ Q(\phi)=e^{\imath m\phi}$.

Separate variable r and θ: $r^2\dfrac{1}{U}\dfrac{d^2U}{dr^2}+\dfrac{1}{P\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\,\dfrac{dP}{d\theta}\right)-\dfrac{m^2}{\sin^2\theta}=0.$

Judicious choice of separation constant: $l(l+1),\quad l=0,1,2,\ldots$

Solutions for $U(r)$: $\dfrac{d^2U}{dr^2}=\dfrac{l(l+1)}{r^2}\,U\quad\Rightarrow\ U(r)=Ar^{l+1}+Br^{-l}$.

ODE for $P(\theta)$: $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\,\dfrac{dP}{d\theta}\right)+\left[l(l+1)-\dfrac{m^2}{\sin^2\theta}\right]P=0.$

The special case $m=0$ of azimuthally symmetrical solutions leads to Legendre polynomials $P(\theta)=P_l(\cos\theta)$ discussed and applied in [lln7].

Generalized Legendre equation (with $x\doteq\cos\theta$):

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0.$$

Solutions with $l=0,1,2,\ldots$ and $|m|\leq l$ are non-divergent for $|x|\leq 1$.

Associated Legendre functions:

- Rodrigues' generator: $P_l^m(x)=\dfrac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\dfrac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$.
- Orthogonality: $\displaystyle\int_{-1}^{+1}dx\,P_{l'}^m(x)P_l^m(x)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{l'l}$.

– Legendre polynomials: $P_l^0(x) = P_l(x)$.

– Case $l = 1$: $P_1^1(x) = -\sqrt{1-x^2}, \quad P_1^0(x) = x, \quad P_1^{-1}(x) = \frac{1}{2}\sqrt{1-x^2}$.

Spherical harmonics combine the angular parts of orthogonal functions in spherical coordinates:

– Definition: $Y_{lm}(\theta,\phi) \doteq \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{\imath m\phi}$.

– Complex conjugate function: $Y_{lm}^*(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi)$.

– Orthonormality: $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y_{l'm'}^*(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm}$.

– Completeness: $\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$.

Expansion of arbitrary angular function:

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}Y_{lm}(\theta,\phi), \quad C_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y_{lm}^*(\theta,\phi) g(\theta,\phi).$$

General solution of Laplace equation expressed in spherical coordinates:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right] Y_{lm}(\theta,\phi).$$