Expansions in Orthogonal Functions [lam12]

Here we revisit separable solutions of the Laplace equations as discussed in [lln7] from a more general perspective.

It is useful to express separable solutions $f(\xi)$ in terms of complete sets of (real or complex) orthonormal functions $U_n(\xi)$:

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi).$$

Key attributes of orthogonal expansion:

- Orthonormality: $\displaystyle\int_a^b d\xi U_n^*(\xi) U_m(\xi) = \delta_{nm}$.
- Completeness: $\displaystyle\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi)$.
- Convergence: $\displaystyle a_n = \int_a^b d\xi U_n^*(\xi) f(\xi)$.

Convergence in the mean minimizes (for given N) the quantity,

$$M_N = \int_a^b d\xi \left| f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right|^2,$$

from which the expression for a_n follows:

▷ $\displaystyle\frac{dM_N}{a_m} = \int_a^b d\xi \left[f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right] U_m^*(\xi) = 0.$

▷ $\displaystyle\int_a^b d\xi \left[U_m^*(\xi) f(\xi) - \sum_{n=1}^{N} a_n U_m^*(\xi) U_n(\xi) \right] = 0.$

▷ $\displaystyle\int_a^b d\xi U_m^*(\xi) f(\xi) - \sum_{n=1}^{N} a_n \delta_{nm} = 0.$

The most familiar applications are Fourier series and Fourier integrals as summarized in the following. A more systematic account of Fourier transform will be presented elsewhere.

Fourier series:

Orthogonal expansion of function $f(x)$ with range $-a/2 \le x \le +a/2$:

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(k_n x) + B_n \sin(k_n x)\right], \quad k_n = \frac{2n\pi}{a}.$$

Orthogonality relations and normalization:

$$\int_{-a/2}^{+a/2} dx \sin(k_m x) \sin(k_n x) = \tfrac{1}{2}a\, \delta_{mn},$$
$$\int_{-a/2}^{+a/2} dx \cos(k_m x) \cos(k_n x) = \tfrac{1}{2}a\, \delta_{mn},$$
$$\int_{-a/2}^{+a/2} dx \sin(k_m x) \cos(k_n x) = 0.$$

Expansion coefficients:

$$A_0 = \frac{2}{a}\int_{-a}^{+a} dx f(x),$$
$$A_n = \frac{2}{a}\int_{-a}^{+a} dx f(x) \cos(k_n x), \quad n = 1, 2, \ldots$$
$$B_n = \frac{2}{a}\int_{-a}^{+a} dx f(x) \sin(k_n x), \quad n = 1, 2, \ldots$$

Simplified circumstances:

- ▷ Symmetric (even) functions: $f(-x) = f(x) \quad \Rightarrow \; B_n = 0.$
- ▷ Antisymmetric (odd) functions: $f(-x) = -f(x) \quad \Rightarrow \; A_n = 0.$

Fourier integral:

Begin with orthonormal set of complex exponentials instead:

$$U_n(x) = \frac{1}{\sqrt{a}} e^{\imath k_n x}, \quad k_n = \frac{2n\pi}{a}, \quad n = 0, \pm 1, \pm 2, \dots$$

Fourier series: $f(x) = \sum_n A_n U_n(x), \quad A_n = \int_{-a/2}^{+a/2} dx U_n^*(x) f(x).$

Othonormality: $\int_{-a/2}^{+a/2} dx U_m^*(x) U_n(x) = \delta_{nm}.$

Completeness: $\sum_n U_n^*(x) U_n(x) = \delta(x - x').$

As the domain widens to infinity, $a \to \infty$, the discrete set of k_n turn into a continuum of infinite width.

The following limits apply:

$$\frac{2\pi n}{a} \longrightarrow k, \quad \sum_n \longrightarrow \int_{-\infty}^{+\infty} dn = \frac{a}{2\pi} \int_{-\infty}^{+\infty} dk, \quad A_n \longrightarrow \sqrt{\frac{2\pi}{a}} A(k).$$

Fourier integral: $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk A(k) e^{\imath kx}$ (inverse transform).

Fourier transform: $A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx f(x) e^{-\imath kx}.$

Orthogonality: $\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx e^{\imath (k-k')x} = \delta(k - k').$

Completeness: $\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{\imath (x-x')k} = \delta(x - x').$