Green's Theorem [lam11]

Green's theorem is a mathematical identity inferred from Gauss's theorem [lln4]. The theorem has proven to be useful for the solution of electrostatic boundary-value problems.

Consider a region of space V bounded by a surface S. Two vector fields, $\phi\nabla\psi$ and $\psi\nabla\phi$, in that region are derived from a pair of scalar fields ϕ, ψ.

We apply an identity for the divergence of the vector fields thus constructed,

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi, \quad \nabla\cdot(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi.$$

Next we integrate the two identities over the region and use Gauss's theorem:

$$\int_V d^3x\big[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\big] = \oint_S da\,\phi\,\frac{\partial\psi}{\partial n}, \tag{1}$$

$$\int_V d^3x\big[\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi\big] = \oint_S da\,\psi\,\frac{\partial\phi}{\partial n}, \tag{2}$$

having set the stage for the surface integration as follows:

$$\phi\nabla\psi\cdot\hat{\mathbf{n}} = \phi\,\frac{\partial\psi}{\partial n}, \quad \psi\nabla\phi\cdot\hat{\mathbf{n}} = \psi\,\frac{\partial\phi}{\partial n}.$$

Green's theorem results from the difference of (1) and (2), which eliminates the gradient product terms:

$$\int_V d^3x\big[\phi\nabla^2\psi - \psi\nabla^2\phi\big] = \oint_S da\left[\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}.\right]. \tag{3}$$

For some applications, Eq. (1) alone is useful by itself and also named Green's theorem. Green's theorem merely transforms a given problem.

In the following we present an application of Green's theorem to electrostatics. Green's theorem, in this case, transforms the Poisson differential equation with given boundary conditions into an integral equation.

We set the two scalar field as follows:

- ▷ $\psi(\mathbf{x}') \doteq \dfrac{1}{|\mathbf{x}-\mathbf{x}'|} = \dfrac{1}{R}$ (distance of field point from source point).
- ▷ $\phi(\mathbf{x}') = \Phi(\mathbf{x}')$ (electrostatic potential).

Then (3) becomes

$$\Rightarrow \int_V d^3x'\left[\Phi(\mathbf{x}')\nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{R}\nabla^2\Phi(\mathbf{x}')\right] = \oint_S da'\left[\Phi\,\frac{\partial}{\partial n'}\frac{1}{R} - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right].$$

Then use the Poisson equation and a mathematical identity,

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}, \quad \nabla^2\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}'),$$

which yields

$$\int_V d^3x' \left[-4\pi\delta(\mathbf{x}-\mathbf{x}')\Phi(\mathbf{x}') + \frac{\rho(\mathbf{x}')}{\epsilon_0 R}\right] = \oint_S da' \left[\Phi\frac{\partial}{\partial n'}\frac{1}{R} - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right].$$

For field points inside the region, $\mathbf{x} \in V$, the integral of the first term is nonzero and we obtain the following equation for the potential Φ:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{4\pi}\oint_S da' \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\frac{1}{R}\right]. \quad (4)$$

- The first surface-integral term represents the electrostatic potential due to a surface-charge density, $\sigma = \epsilon_0 \partial\Phi/\partial n'$.
- The second surface-integral term represents the electrostatic potential due to an electric dipole layer, $D = -\epsilon_0\Phi$.

For field points outside the region, $\mathbf{x} \notin V$, the integral of the first term vanishes. We can write

$$\frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{4\pi}\oint_S da' \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\frac{1}{R}\right] = 0.$$

Any electric potential generated by the charge distribution is undone at the surface of the region by a combination of discontinuities in electric field (first boundary term) and discontinuities in potential (second boundary term).

In the current context, it is useful to introduce the Green's function,[1]

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + F(\mathbf{x},\mathbf{x}'). \quad (5)$$

Its a solution of the equation,

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'),$$

if the function $F(\mathbf{x},\mathbf{x}')$ satisfies the Laplace equation,

$$\nabla'^2 F(\mathbf{x},\mathbf{x}') = 0,$$

in the region V bounded by surface S.

[1]Green's functions are more systematically introduced elsewhere. They are solutions of $\mathcal{L}G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$, where $\mathcal{L}$ is a linear partial differential operator. The solution of the linear partial differential equation, $\mathcal{L}\psi(\mathbf{x}) = f(\mathbf{x})$, with given $f(\mathbf{x})$ can then be expressed as the integral $\psi(\mathbf{x}) = \int d^3x' G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')$.

The use of Green's theorem with $\psi(\mathbf{x}') = G(\mathbf{x},\mathbf{x}')$ and $\phi(\mathbf{x}') = \Phi(\mathbf{x}')$ yields a generalization of the integral equation (4) derived earlier:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}') + \frac{1}{4\pi}\oint_S da'\left[G(\mathbf{x},\mathbf{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\mathbf{x}')\,\frac{\partial G}{\partial n'}\right]. \qquad (6)$$

In an application to a situation with Dirichlet boundary conditions, we must, in essence, solve the Laplace equation for $F(\mathbf{x},\mathbf{x}')$ with modified boundary conditions, namely such that

$$G(\mathbf{x},\mathbf{x}') = 0 \quad : \; \mathbf{x}' \in S.$$

The solution of the boundary-value problem then becomes,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V d^3x'\rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}') - \frac{1}{4\pi}\oint_S da'\Phi(\mathbf{x}')\,\frac{\partial G}{\partial n'}. \qquad (7)$$

If the boundary S is moved to infinity in all direction the the last term of (6) approaches zero and the Green's function (5) reduces to its first term.

Applications to situations with Neumann boundary conditions work similarly albeit with some complications mixed in [Jackson 1999].