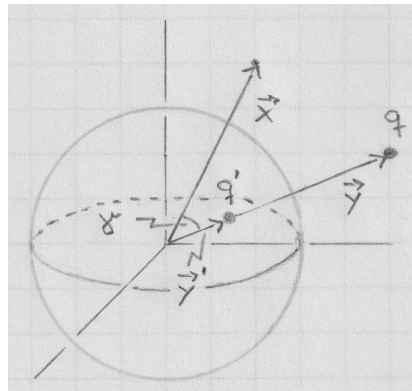
Electric Potential of Point Charge Near Conducting Sphere [lam10]

Application of the method of image charges:

The method of image charges was introduced in [lln6]. This is a more advanced application.

- Sphere of radius a centered at origin of the coordinate system.
- Point charge q at $\mathbf{y} = y\mathbf{n}'$ outside sphere.
- Field point at $\mathbf{x} = x\mathbf{n}$ outside sphere.
- Image charge q' at $\mathbf{y}' = y'\mathbf{n}'$ inside sphere.



Electric potential: $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\epsilon_0}\left[\dfrac{q}{|x\mathbf{n} - y\mathbf{n}'|} + \dfrac{q'}{|x\mathbf{n} - y'\mathbf{n}'|}\right].$

Grounded-sphere condition: $\Phi(\mathbf{x})\Big|_{x=a} = 0$

$$\Rightarrow \ \frac{q}{a}\frac{1}{|\mathbf{n} - (y/a)\mathbf{n}'|} + \frac{q'}{y'}\frac{1}{|\mathbf{n}' - (a/y')\mathbf{n}|} = 0 \quad \Rightarrow \ \frac{q}{a} = -\frac{q'}{y'}, \quad \frac{y}{a} = \frac{a}{y'}.$$

Resulting image charge specifications: $q' = -\dfrac{a}{y}\,q, \quad y' = \dfrac{a^2}{y}.$

When the point charge q approaches the surface from the outside, the image charge q' approaches it from the inside. Very near the spherical surface, the two charges are close to equal in magnitude and close to equal in distance.

The electric potential depends on two distances and one angle, $\cos\gamma = \mathbf{n}\cdot\mathbf{n}'$:

$$\Phi(x,y,\gamma) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 - 2xy\cos\gamma}} - \frac{a/y}{\sqrt{x^2 + a^4/y^2 - 2(xa^2/y)\cos\gamma}}\right].$$

Surface charge density on the sphere:

$$\sigma(y,\gamma) = -\epsilon_0\frac{\partial\Phi}{\partial x}\Big|_{x=a} = -\frac{q}{4\pi a^2}\frac{(a/y)^3[1 - (a/y)^2]}{\left[1 + (a/y)^2 - 2(a/y)\cos\gamma\right]^{3/2}}.$$

The surface charge density integrated over the sphere equals the image charge:

$$q' = \int_0^\pi d\gamma(2\pi a)(a\sin\gamma)\sigma(y,\gamma) = -\frac{a}{y}\,q.$$

Force between point charge and image charge:

Force between point charge and grounded sphere (or between q and q'):

$$F = \frac{1}{4\pi\epsilon_0}\frac{|qq'|}{(y-y')^2} = \frac{q^2}{4\pi\epsilon_0 a^2}\frac{(a/y)^3}{\left[1-(a/y)^2\right]^2} \stackrel{y\gg a}{\longrightarrow} \frac{1}{4\pi\epsilon_0}\frac{aq^2}{y^3}.$$

The attribute $qq' < 0$ makes this force attractive at all distances. It is the force between a charge q and an induced electric dipole.

The induced electric dipole moment depends on the distance as follows:

$$p = \int_0^\pi d\gamma (2\pi a)(a\sin\gamma)\sigma(y,\gamma)(a\cos\gamma) = -\frac{a^3}{y^2}\,q.$$

Force between point charge and charged conducting sphere:

Here we replace the grounded sphere by a sphere with charge Q. The image charge, $q' = -(a/y)q$, ensures vanishing potential on the surface. The remainder, $Q - q'$, then spreads uniformly across the surface.

The electric potential from the previous case thus acquires one extra term:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{a}{y}\frac{1}{|\mathbf{x}-(a^2/y^2)\mathbf{y}|} + \frac{Q/q + a/y}{|\mathbf{x}|}\right].$$

The Coulomb force also acquires an extra term. The resulting expression, valid for $y > a$, can be simplified as follows:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\left[\frac{q[Q+(a/y)q]}{y^2} + \frac{q^2}{a^2}\frac{(a/y)^3}{\left[1-(a/y)^2\right]^2}\right]\mathbf{n}$$
$$= \frac{qQ}{4\pi\epsilon_0 y^2}\left[1 - \frac{q}{Q}\left(\frac{a}{y}\right)^3\frac{2-(a/y)^2}{[1-(a/y)^2]^2}\right]\mathbf{n}.$$

Unsurprisingly, this force is attractive at all distances if $qQ < 0$ and stronger than the force between two point charges Q and q at distance y.

The force changes direction with distance if $qQ > 0$. It is attractive at short distance and repulsive at long distance.

Attraction at very short distance from the surface of a conductor stabilizes mobile charge carriers against escape. Thermionic emission of electrons from the surface of conductors is subject to an energy barrier (Schottky effect).