Generalized Functions

Dirac delta function:

The generalized function $\delta(x)$, also known as distribution, is defined by the following attributes:

$$\delta(x) = \begin{cases} 0 & : \; x \neq 0, \\ \infty & : \; x = 0, \end{cases} \qquad \int_{-\infty}^{+\infty} dx \, \delta(x) = 1.$$

Some consequences and generalizations:

$\triangleright \; \displaystyle\int_{-\infty}^{+\infty} dx \, f(x)\delta(x-a) = f(a).$

$\triangleright \; \displaystyle\int_{-\infty}^{+\infty} dx \, f(x)\delta(bx) = \frac{1}{|b|} f(0).$

$\triangleright \; \displaystyle\int_{-\infty}^{+\infty} dx \, f(x)\delta(bx-a) = \frac{1}{|b|} f(a/b).$

$\triangleright \; \displaystyle\int_{-\infty}^{+\infty} dx \, f(x)\delta\big(g(x)\big) = \sum_i \frac{1}{|g'(x_i)|} f(x_i); \quad g(x_i) = 0, \quad x = x_i.$

$\triangleright \; \delta(\mathbf{x} - \mathbf{a}) = \delta(x - a_x)\delta(y - a_y)\delta(z - a_z).$