Coordinate Systems [gmd2]

Curvilinear coordinates:

Here we introduce a notation that is applicable, in equal measure, to Cartesian, cylindrical, and spherical coordinates in 3-dimensional space. The basic ingredients (with $i = 1, 2, 3$ or equivalent) are

- ▷ $\hat{\mathbf{e}}_i$: mutually orthogonal unit vectors,
- ▷ h_i: scale factors,
- ▷ du_i: coordinate increments.

Infinitesimal displacement: $d\mathbf{s} = h_1 du_1 \hat{\mathbf{e}}_1 + h_2 du_2 \hat{\mathbf{e}}_2 + h_3 du_3 \hat{\mathbf{e}}_3$.

Construction of gradient:

$$df = \nabla f \cdot d\mathbf{s} = \sum_i (\nabla f)_i h_i du_i = \sum_i \frac{\partial f}{\partial u_i} \quad \Rightarrow \quad (\nabla f)_i = \frac{1}{h_i}\frac{\partial f}{\partial u_i}.$$

Construction of divergence:

$$(\nabla \cdot \mathbf{F})dV = \mathbf{F} \cdot d\mathbf{A}, \quad dV = \prod_i h_i du_i, \quad dA_i = h_j u_j h_k u_k.$$

$$\Rightarrow \quad \nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \sum_{\{ijk\}} \frac{\partial}{\partial u_i}(F_i h_j h_k), \quad \{ijk\} = \mathrm{cycl}\{123\}.$$

Construction of curl:

$$\sum_i (\nabla \times \mathbf{F})_i dA_i = \sum_i F_i dl_i, \quad dl_i = h_i du_i.$$

$$\Rightarrow \quad (\nabla \times \mathbf{F})_k = \frac{1}{h_i h_j}\left[\frac{\partial}{\partial u_i}(F_j h_j) - \frac{\partial}{\partial u_j}(F_i h_i)\right], \quad \{ijk\} = \mathrm{cycl}\{123\}.$$

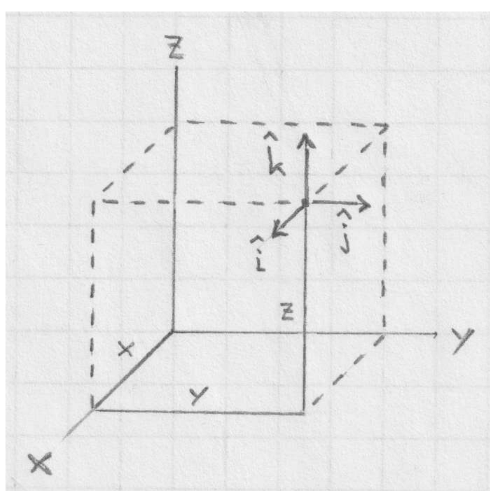
$$\Rightarrow \quad \nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\hat{\mathbf{e}}_1}{h_2 h_3} & \dfrac{\hat{\mathbf{e}}_2}{h_3 h_1} & \dfrac{\hat{\mathbf{e}}_3}{h_1 h_2} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

Construction of Laplacian:

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_1 h_2 h_3} \sum_{\{ijk\}} \frac{\partial}{\partial u_i}\left(\frac{h_j h_k}{h_i}\frac{\partial f}{\partial u_i}\right).$$

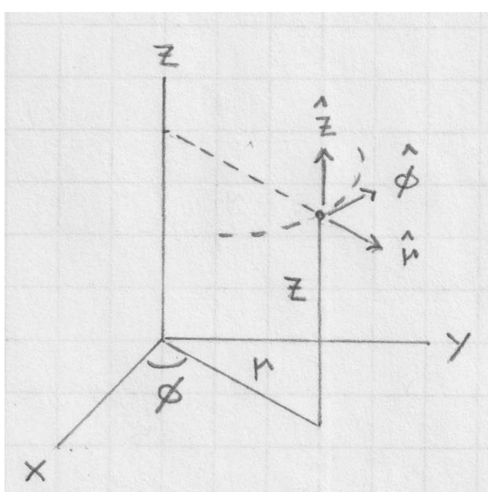
Cartesian coordinates:

position	$\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$
displacement	$d\mathbf{s} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$
scale factors	$h_x = 1,\; h_y = 1,\; h_z = 1$
volume element	$dV = dx\,dy\,dz$
area elements	$dA_x = dy\,dz,\; dA_y = dx\,dz,\; dA_z = dx\,dy$
line elements	$dl_x = dx,\; dl_y = dy,\; dl_z = dz$
gradient	$\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}$
divergence	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
curl	$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$



Cylindrical coordinates:

position	$\mathbf{x} = r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}$
displacement	$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$
scale factors	$h_r = 1,\ h_\phi = r,\ h_z = 1$
volume element	$dV = r dr\,d\phi\,dz$
area elements	$dA_r = r d\phi\,dz,\ dA_\phi = dr\,dz,\ dA_z = r dr\,d\phi$
line elements	$dl_r = dr,\ dl_\phi = r\,d\phi,\ dl_z = dz$
gradient	$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}$
divergence	$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (r F_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$
Laplacian	$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$
curl	$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\,\hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\,\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial (r F_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right)\,\hat{\mathbf{z}}$



Spherical coordinates:

position	$\mathbf{x} = r\,\hat{\mathbf{r}}$
displacement	$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$
scale factors	$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$
volume element	$dV = r^2 dr\sin\theta\,d\theta\,d\phi$
area elements	$dA_r = r^2\sin\theta\,d\theta\,d\phi,\ dA_\theta = r dr\sin\theta\,d\phi,\ dA_\phi = r dr\,d\theta,$
line elements	$dl_r = dr,\ dl_\theta = r\,d\theta,\ dl_\phi = r\sin\theta\,d\phi$
gradient	$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\,\hat{\boldsymbol{\phi}}$
divergence	$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\sin\theta F_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$
Laplacian	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
curl	$\nabla\times\mathbf{F} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,F_\phi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right)\hat{\mathbf{r}}$
	$+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(r\,F_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$