Vector Analysis [gmd1]

Vectors are quantities with magnitude and direction. Vector algebra is about addition and multiplications of vectors. Vector analysis also includes differential and integral operations with vectors.

The theory of electromagnetism heavily relies on vector analysis. Here we compile a set of tools without proofs for use throughout the course.

Vectors in 3-dimensional space have three components. We consider three coordinate systems: Cartesian, cylindrical, and spherical coordinates.

Expression with vectors have a different look in different coordinate systems. The choice of coordinate system is informed by symmetry. There are coordinate-independent expressions, named geometric representations.

Vector addition:

Cartesian unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.

Vector in Cartesian components: $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} = (A_x, A_y, A_z)$.

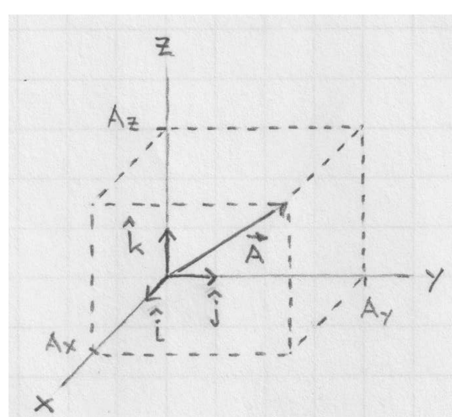
Magnitude: $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$.

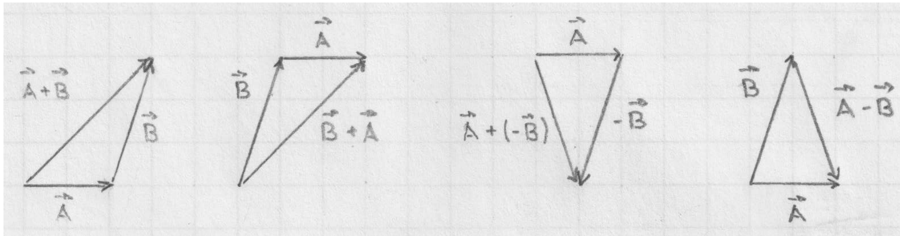
Null vector, $\mathbf{0} = (0, 0, 0)$, has zero magnitude and no direction.

Sum: $\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$.

Vector addition is commutative and associative. The vectors $\mathbf{A}$ and $-\mathbf{A}$ have the same magnitude and opposite direction. Subtraction of $\mathbf{B}$ means addition of $-\mathbf{B}$. A vector multiplied by a scalar a (real number) changes its magnitude if $a \neq \pm 1$ and switches its direction if $a < 0$.

- ▷ $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
- ▷ $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.
- ▷ $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.
- ▷ $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$.
- ▷ $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$.
- ▷ $(ab)\mathbf{A} = a(b\mathbf{A})$.





Dot product of vectors:

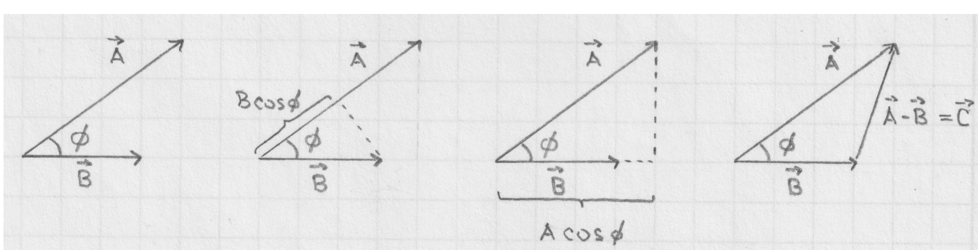
The dot product yields a scalar. It is also named scalar product. The dot product is commutative. Perpendicular vectors yield zero. Parallel vectors yield the product of magnitudes.

The magnitude of a vector, the angle between two vectors, and the law of cosines can be inferred from dot products.

- ▷ $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.
- ▷ $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$.
- ▷ $\mathbf{A} \cdot (c\mathbf{B}) = c(\mathbf{A} \cdot \mathbf{B})$.
- ▷ $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$.
- ▷ $\mathbf{A} \cdot \mathbf{B} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \cdot (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) = A_xB_x + A_yB_y + A_zB_z$.
- ▷ $A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (magnitude).
- ▷ $\mathbf{A} \cdot \mathbf{B} = AB\cos\phi$.
- ▷ $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} - 2\mathbf{A} \cdot \mathbf{B} = A^2 + B^2 - 2AB\cos\phi$.

The last relation is illustrated by the triangle on the right.

Law of cosines: $C^2 = A^2 + B^2 - 2AB\cos\phi$.

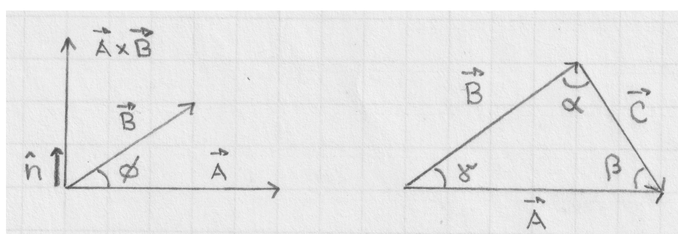


Cross product of vectors:

The cross product yields a vector. It is also named vector product. Switching the sequence factors switches the direction of the product. Parallel or antiparallel vectors yield the null vector. The direction of the product is perpendicular to the plane spanned by the factors.

- ▷ $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.
- ▷ $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.
- ▷ $\mathbf{A} \times (c\mathbf{B}) = (c\mathbf{A}) \times \mathbf{B} = c(\mathbf{A} \times \mathbf{B})$.
- ▷ $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0, \quad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$.
- ▷ $\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$.
- ▷ $\mathbf{A} \times \mathbf{B} = AB \sin\phi\, \hat{\mathbf{n}} \quad (\hat{\mathbf{n}} \perp \mathbf{A},\ \hat{\mathbf{n}} \perp \mathbf{B}$, right-hand rule)
- ▷ $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$.

The area of the parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ is equal to $|\mathbf{A} \times \mathbf{B}|$.



The law of sines is derived from three equivalent expressions of the area of the triangle on the right:

$$\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \frac{1}{2}|\mathbf{B} \times \mathbf{C}| = \frac{1}{2}|\mathbf{C} \times \mathbf{A}| \quad \Rightarrow \ AB\sin\gamma = BC\sin\alpha = CA\sin\beta.$$

$$\Rightarrow \ \frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C} \quad \text{(law of sines)}.$$

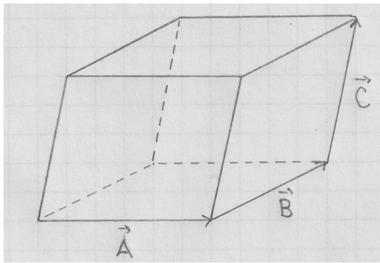
Identities involving products of vectors:

Triple scalar product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$.

This product yields a scalar from three vectors. It is invariant under cyclic permutation of the factors.

Geometrically, if the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, are the sides of a parallelepiped, then the triple scalar product is $\pm$ its volume:

$$\pm V = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$



Three mutually orhtogonal vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ form a right-handed triad if $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) > 0$. This condition is satisfield by the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.

Triple vector product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

This product yields a vector which must be perpendicular to $\mathbf{B} \times \mathbf{C}$, implying that is parallel to the plane spanned by $\mathbf{B}$ and $\mathbf{C}$. The second expression is indeed an expansion in that basis.

Further useful identities involving multiple vector or scalar products are the following:

- ▷ $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$.
- ▷ $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{D})\mathbf{C} - (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})\mathbf{D}$.

Differential operators:

Position vector: $\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.

Scalar function: $f(\mathbf{x}) = f(x, y, z)$.

Vector function: $\mathbf{F}(\mathbf{x}) = F_x(x, y, z)\,\hat{\mathbf{i}} + F_y(x, y, z)\,\hat{\mathbf{j}} + F_z(x, y, z)\,\hat{\mathbf{k}}$.

Differential operator:[1] $\nabla = \dfrac{\partial}{\partial x}\,\hat{\mathbf{i}} + \dfrac{\partial}{\partial y}\,\hat{\mathbf{j}} + \dfrac{\partial}{\partial z}\,\hat{\mathbf{k}}$.

Gradient: $\nabla f = \dfrac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \dfrac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \dfrac{\partial f}{\partial z}\,\hat{\mathbf{k}}$.

Divergence: $\nabla \cdot \mathbf{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$.

Curl: $\nabla \times \mathbf{F} = \left(\dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$.

Laplacian: $\nabla^2 f = \nabla \cdot (\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$.

These expressions are applicable in Cartesian coordinate systems. The gradient is applied to a scalar function and produces a vector function. The opposite holds for the divergence.

The curl produces a vector function from a vector function and the Laplacian a scalar function from a scalar function. The latter can be applied to the (Cartesian) components of a vector, producing another vector.

[1] The symbol of the operator ∇ is named ‘nabla’. When used as a gradient, divergence, or curl, we say ‘’del f’, ‘del dot $\mathbf{F}$’, and ‘del cross $\mathbf{F}$’, respectively.

Identities involving differential operators:

Differential operators: gradient, divergence, curl, Laplacian.

Scalar functions: $f(\mathbf{x})$, $g(\mathbf{x})$.

Vector functions: $\mathbf{F}(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$.

Derivatives of products:[2]

- $\triangleright$ $\nabla(fg) = f(\nabla g) + g(\nabla f)$
- $\triangleright$ $\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$
- $\triangleright$ $\nabla\cdot(g\mathbf{F}) = g(\nabla\cdot\mathbf{F}) + (\nabla g)\cdot\mathbf{F}$
- $\triangleright$ $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})$
- $\triangleright$ $\nabla\times(g\mathbf{F}) = g(\nabla\times\mathbf{F}) + (\nabla g)\times\mathbf{F}$
- $\triangleright$ $\nabla\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F})$

Derivatives of quotients:

- $\triangleright$ $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g(\nabla f) - f(\nabla g)}{g^2}$
- $\triangleright$ $\nabla\cdot\left(\dfrac{\mathbf{F}}{g}\right) = \dfrac{g(\nabla\cdot\mathbf{F}) - \mathbf{F}\cdot(\nabla g)}{g^2}$
- $\triangleright$ $\nabla\times\left(\dfrac{\mathbf{F}}{g}\right) = \dfrac{g(\nabla\times\mathbf{F}) - \mathbf{F}\times(\nabla g)}{g^2}$

Products of derivatives:[3]

- $\triangleright$ $\nabla\cdot\nabla f = \nabla^2 f$
- $\triangleright$ $\nabla\times(\nabla f) = 0$
- $\triangleright$ $\nabla\cdot(\nabla\times\mathbf{F}) = 0$
- $\triangleright$ $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$

Vectors ∇f are named *irrotational* and have zero curl.

Vectors $\nabla\times\mathbf{F}$ are named *solenoidal* and have zero divergence.

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[2] In the expression $(\mathbf{F}\cdot\nabla)\mathbf{G}$, the scalar operator $\mathbf{F}\cdot\nabla = F_x(\partial/\partial x) + F_y(\partial/\partial y) + F_z(\partial/\partial z)$ acts on the vector $\mathbf{G}$ and thus yields a vector.

[3] The Laplacian operating on a vector, $\nabla^2\mathbf{F}$, has a straightforward meaning for Cartesian components: it acts on each component of $\mathbf{F}$ to produce the components of $\nabla^2\mathbf{F}$. For curvilinear coordinates, the last identity can be used as the definition of $\nabla^2\mathbf{F}$.

Differentials of scalars:

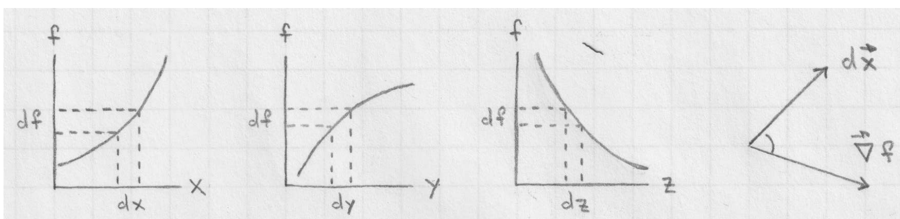
The differential of a scalar function $f(\mathbf{x})$ is constructed as a dot product from two vectors: its gradient and an infinitesimal displacement vector:

$$df = d\mathbf{x} \cdot \nabla f, \quad d\mathbf{x} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}.$$

Here the scalar f is acted on by the scalar operator,

$$d\mathbf{x} \cdot \nabla = dx\,\frac{\partial}{\partial x} + dy\,\frac{\partial}{\partial y} + dz\,\frac{\partial}{\partial x},$$

which the scalar df. If $d\mathbf{x}$ and ∇f are perpendicular to each other, $d\mathbf{x}$ is along a line of constant $f(\mathbf{x})$. The vector ∇f points in the direction of steepest ascent of $f(\mathbf{x})$.



The differential df thus derived is named *exact* differential. A more general differential has the form,

$$dg = g_x(\mathbf{x})dx + g_y(\mathbf{x})dy + g_z(\mathbf{x})dz,$$

with arbitrary functions g_x, g_y, g_z in the role of coefficients in the differential. Such a differential is, in general, *inexact*.

A set of coefficients g_x, g_y, g_z specify an exact differential if they are the components of an irrotational vector, $\nabla \times \mathbf{g} = 0$, which implies the conditions,

$$\frac{\partial g_y}{\partial x} = \frac{\partial g_x}{\partial y}, \quad \frac{\partial g_z}{\partial y} = \frac{\partial g_y}{\partial z}, \quad \frac{\partial g_x}{\partial z} = \frac{\partial g_z}{\partial x}.$$

Any irrotational vector can be expressed as the gradient of a scalar, $\mathbf{g} = \nabla f$, implying the conditions,

$$g_x = \frac{\partial f}{\partial x}, \quad g_y = \frac{\partial f}{\partial y}, \quad g_z = \frac{\partial f}{\partial z}.$$

The difference between exact and inexact differentials matters for their integration between points or around a loop in space (a later topic).

Differentials of vectors:

The differential of a vector function $\mathbf{F}(\mathbf{x})$ is constructed as follows:

$$d\mathbf{F} = \mathbf{F}(\mathbf{x} + d\mathbf{x}) - \mathbf{F}(\mathbf{x}) = (d\mathbf{x} \cdot \nabla)\mathbf{F}.$$

Here the vector $\mathbf{F}$ is acted on by the same scalar operator,

$$d\mathbf{x} \cdot \nabla = dx\,\frac{\partial}{\partial x} + dy\,\frac{\partial}{\partial y} + dz\,\frac{\partial}{\partial x},$$

which produces the vector,

$$\begin{aligned} d\mathbf{F} = \left(dx\,\frac{\partial F_x}{\partial x} + dy\,\frac{\partial F_x}{\partial y} + dz\,\frac{\partial F_x}{\partial z}\right)\hat{\mathbf{i}} + \left(dx\,\frac{\partial F_y}{\partial x} + dy\,\frac{\partial F_y}{\partial y} + dz\,\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{j}} \\ + \left(dx\,\frac{\partial F_z}{\partial x} + dy\,\frac{\partial F_z}{\partial y} + dz\,\frac{\partial F_z}{\partial z}\right)\hat{\mathbf{k}} \end{aligned}$$

The differential of position $\mathbf{x}$ simplifies into $d\mathbf{x} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$.

The differential of an irrotational vector, $\nabla \times \mathbf{F} = 0$, can be rewritten as

$$\begin{aligned} d\mathbf{F} &= \left(dx\,\frac{\partial F_x}{\partial x} + dy\,\frac{\partial F_y}{\partial x} + dz\,\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{i}} + \left(dx\,\frac{\partial F_x}{\partial y} + dy\,\frac{\partial F_y}{\partial y} + dz\,\frac{\partial F_z}{\partial y}\right)\hat{\mathbf{j}} \\ &\qquad + \left(dx\,\frac{\partial F_x}{\partial z} + dy\,\frac{\partial F_y}{\partial z} + dz\,\frac{\partial F_z}{\partial z}\right)\hat{\mathbf{k}} \\ &= \left(d\mathbf{x} \cdot \frac{\partial}{\partial x}\,\mathbf{F}\right)\hat{\mathbf{i}} + \left(d\mathbf{x} \cdot \frac{\partial}{\partial y}\,\mathbf{F}\right)\hat{\mathbf{j}} + \left(d\mathbf{x} \cdot \frac{\partial}{\partial z}\,\mathbf{F}\right)\hat{\mathbf{k}} \\ &= dx\nabla F_x + dy\nabla F_y + dz\nabla F_z. \end{aligned}$$

Line integrals:

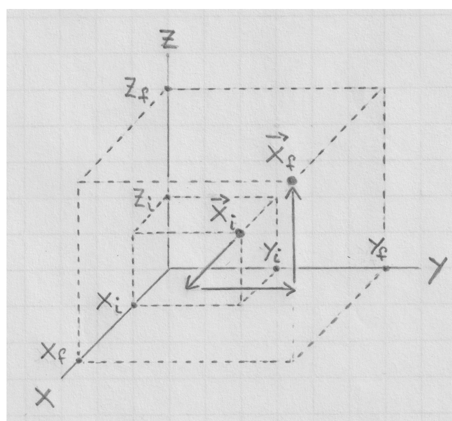
Integration along a path C in space is called a line integral. Line integrals along closed paths are named loop integrals.

The line integral of a scalar function $f(\mathbf{x})$ yields a vector $\mathbf{I}$:

$$\begin{aligned}\mathbf{I} &= \int_C d\mathbf{x} f(\mathbf{x}) \\ &= \int_{x_i}^{x_f} dx f(x, y_i, z_i)\,\hat{\mathbf{i}} + \int_{y_i}^{y_f} dy f(x_f, y, z_i)\,\hat{\mathbf{j}} + \int_{z_i}^{z_f} dz f(x_f, y_f, z)\,\hat{\mathbf{k}}.\end{aligned}$$

The second expression holds for the specific path shown with initial point $\mathbf{x}_i = (x_i, y_i, z_i)$ and final point $\mathbf{x}_f = (x_f, y_f, z_f)$. In general, the result depends on the path chosen.

If $f(\mathbf{x})$ is a constant, the integral is path-independent. For $f \equiv 1$ the integral becomes the distance vector between the endpoints of the path: $\mathbf{I} = \mathbf{x}_f - \mathbf{x}_i$.



The line integral of a vector function $\mathbf{F}(\mathbf{x})$ yields a scalar I if it is constructed as follows:

$$\begin{aligned}I &= \int_C d\mathbf{x} \cdot \mathbf{F}(\mathbf{x}) \\ &= \int_{x_i}^{x_f} dx F_x(x, y_i, z_i) + \int_{y_i}^{y_f} dy F_y(x_f, y, z_i) + \int_{z_i}^{z_f} dz F_z(x_f, y_f, z).\end{aligned}$$

The second expression holds for the same specific path shown above. In general, the result again depends on the path chosen.

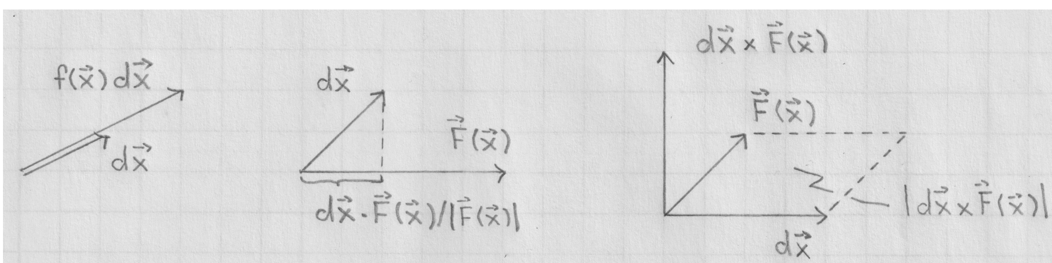
The integral is path-independent if the vector is irrotational, $\nabla \times \mathbf{F} = 0$, in which case it is the gradient of a scalar, $\mathbf{F} = \nabla f$. This implies that the differential df of then scalar function $f(\mathbf{x})$ is exact.

For this case, the integral along the path chosen becomes

$$\begin{aligned} I &= \int_{x_i}^{x_f} dx \frac{\partial f}{\partial x}\Big|_{y_i,z_i} + \int_{y_i}^{y_f} dy \frac{\partial f}{\partial y}\Big|_{x_f,z_i} + \int_{z_i}^{z_f} dz \frac{\partial f}{\partial z}\Big|_{x_f,y_f} \\ &= [f(x_f, y_i, z_i) - f(x_i, y_i, z_i)] + [f(x_f, y_f, z_i) - f(x_f, y_i, z_i)] \\ &\qquad + [f(x_f, y_f, z_f) - f(x_f, y_f, z_i)] \\ &= f(x_f, y_f, z_f) - f(x_i, y_i, z_i) = f(\mathbf{x}_i) - f(\mathbf{x}_f). \end{aligned}$$

The line integral of a vector function $\mathbf{F}(\mathbf{x})$ yields a vector $\mathbf{I}$ if it is constructed as cross product instead of a dot product:

$$\mathbf{I} = \int_C d\mathbf{x} \times \mathbf{F}(\mathbf{x}).$$



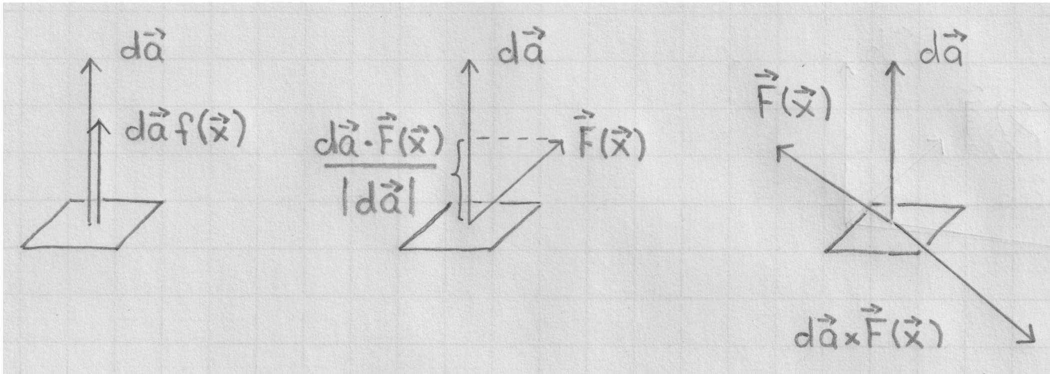
Surface integrals:

The centerpiece of surface integrals is the vector $d\mathbf{a}$ associated with elements of surface area. It is chosen sufficiently small to become essentially flat on the surface in question and directed perpendicular to that plane.

Surface integrals defined as follows then produce a vector, a scalar, and a vector again from left to right.

$$\mathbf{I} = \int_S d\mathbf{a} f(\mathbf{x}), \quad I = \int_S d\mathbf{a} \cdot \mathbf{F}(\mathbf{x}), \quad \mathbf{I} = \int_S d\mathbf{a} \times \mathbf{F}(\mathbf{x}).$$

In the case of an open surface S, one of two options must be chosen for the direction of the area vector. Closed surfaces have an inside and an outside. The convention here is that $d\mathbf{a}$ points toward the outside.



Note that each area element contributes a vector perpendicular to the surface in the first integral and a vector tangential to the surface in the third integral.

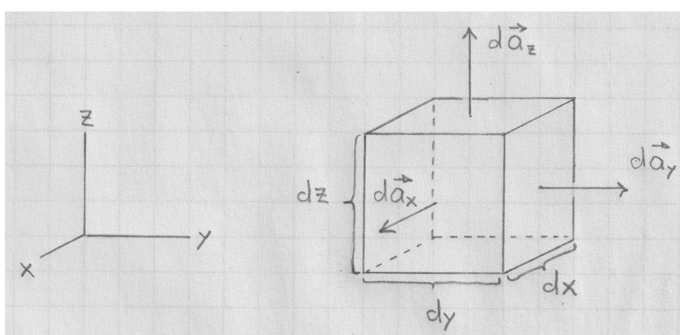
The second integral, which is a scalar, is the most widely occurring in physics applications. Here the scalar I represents the flux associated with the field $\mathbf{F}(\mathbf{x})$ for the surface S.

For completeness we note that the volume integral of a scalar (vector) function yields a scalar (vector) quantity.

Differential operators from integrals:

A cube of infinitesimal volume $V = dxdydz$ is postioned at $\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Its six faces have area vectors,

$$\pm d\mathbf{a}_x = \pm dydz\,\hat{\mathbf{i}}, \quad \pm d\mathbf{a}_y = \pm dxdz\,\hat{\mathbf{j}}, \quad \pm d\mathbf{a}_z = \pm dxdy\,\hat{\mathbf{k}}.$$



The integral of a scalar function $f(\mathbf{x})$ or a vector function $\mathbf{F}(\mathbf{x})$ over the surface of this cube combined with the limit $V \to 0$ can be used to reproduce the gradient, divergence, and curl operators in Cartesian components:

$$\nabla f = \lim_{V\to 0} \frac{1}{V} \oint_S d\mathbf{a}\, f, \quad \nabla\cdot\mathbf{F} = \lim_{V\to 0} \frac{1}{V} \oint_S d\mathbf{a}\cdot\mathbf{F}, \quad \nabla\times\mathbf{F} = \lim_{V\to 0} \frac{1}{V} \oint_S d\mathbf{a}\times\mathbf{F}.$$

We have noted earlier that the first integral yields a vector. We can evaluate the surface integral by using differentials of f and the above area elements:

$$\frac{1}{V}\oint_S d\mathbf{a}\, f \rightsquigarrow \frac{1}{dxdydz}\left[dydz\frac{\partial f}{\partial x}dx\,\hat{\mathbf{i}} + dxdz\frac{\partial f}{\partial y}dy\,\hat{\mathbf{j}} + dxdy\frac{\partial f}{\partial z}dz\,\hat{\mathbf{k}}\right]$$

$$\Rightarrow\ \nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}.$$

For the evaluation of the second integral, which yields a scalar, we use differentials of the components of $\mathbf{F}$ and the above area vectors in combination with the elementary dot products,

$$\hat{\mathbf{i}}\cdot\mathbf{F} = F_x, \quad \hat{\mathbf{j}}\cdot\mathbf{F} = F_y, \quad \hat{\mathbf{k}}\cdot\mathbf{F} = F_z.$$

We can thus write.

$$\frac{1}{V}\oint_S d\mathbf{a}\cdot\mathbf{F} \rightsquigarrow \frac{1}{dxdydz}\left[dydz\frac{\partial F_x}{\partial x}dx + +dxdz\frac{\partial F_y}{\partial y}dy + dxdy\frac{\partial F_z}{\partial z}dz\right]$$

$$\Rightarrow\ \nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The third integral, which yields a vector, is evaluated along the same lines. Here we use the elementary cross products,

$$\hat{\mathbf{i}}\times\mathbf{F} = F_y\,\hat{\mathbf{k}} - F_z\,\hat{\mathbf{j}}, \quad \hat{\mathbf{j}}\times\mathbf{F} = F_z\,\hat{\mathbf{i}} - F_x\,\hat{\mathbf{k}}, \quad \hat{\mathbf{k}}\times\mathbf{F} = F_x\,\hat{\mathbf{j}} - F_y\,\hat{\mathbf{i}},$$

and write,

$$\begin{aligned}\frac{1}{V}\oint_S d\mathbf{a}\times\mathbf{F} \rightsquigarrow \frac{1}{dxdydz}\Bigg[&dxdy\left(\frac{\partial F_y}{\partial x}\,\hat{\mathbf{k}} - \frac{\partial F_z}{\partial x}\,\hat{\mathbf{j}}\right)dx \\ &+ dxdz\left(\frac{\partial F_z}{\partial y}\,\hat{\mathbf{i}} - \frac{\partial F_x}{\partial y}\,\hat{\mathbf{k}}\right)dy + dydz\left(\frac{\partial F_x}{\partial z}\,\hat{\mathbf{j}} - \frac{\partial F_y}{\partial z}\,\hat{\mathbf{i}}\right)dz\Bigg].\end{aligned}$$

$$\Rightarrow\ \nabla\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}.$$

Integral theorems:

▷ *Gauss's theorem* (also named divergence theorem) relates the flux of the vector field $\mathbf{F}$ through the closed surface S to the integral of the scalar quantity $\nabla \cdot \mathbf{F}$ over the (unique) interior volume V:

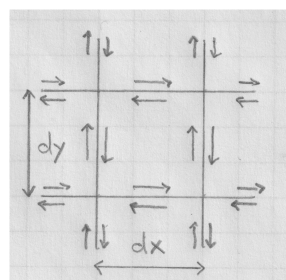
$$\oint_S d\mathbf{a} \cdot \mathbf{F} = \int_V d^3x \, \nabla \cdot \mathbf{F}.$$

The proof can be constructed by dividing the volume V into infinitesimal cubes and apply the second integral of the previous section. The surface integrals over all interior walls cancel.

▷ *Stokes' theorem* relates the circulation of vector field $\mathbf{F}$ along the loop (closed path) C to the integral of the scalar flux quantity constructed from the vector $\nabla \times \mathbf{F}$ and the element $d\mathbf{a}$ of an open surface S with perimeter C:

$$\oint_C d\mathbf{l} \cdot \mathbf{F} = \int_S d\mathbf{a} \cdot (\nabla \times \mathbf{F}).$$

The surface integral is unique, even though the surface S for given perimeter C is not. The proof for a flat surface (in the xy-plane) can be constructed by dividing the surface into infinitesimal squares. The loop integral around each square of area $da = dxdy$ evaluated by use of the differential of $\mathbf{F}$ becomes



$$\oint_{\square} d\mathbf{l} \cdot \mathbf{F} = \int_y^{y+dy} dy[F_y(x+dx, y) - F(x,y)] + \int_x^{x+dx} dx[F_x(x, dy) - F(x, y+dy)]$$

$$\rightsquigarrow dy \, \frac{\partial F_y}{\partial x} \, dx - dx \, \frac{\partial F_x}{\partial y} \, dy = da \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = d\mathbf{a} \cdot (\nabla \times \mathbf{F})_z$$

All line integrals over interior sides of squares cancel.

▷ The *gradient theorem* is an integral version of the identity $\nabla \times (\nabla f) = 0$, stated in two versions as follows:

$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} d\mathbf{x} \cdot \nabla f = f(\mathbf{x}_1) - f(\mathbf{x}_0) \quad \text{(independent of the path chosen)},$$

$$\oint d\mathbf{x} \cdot \nabla f = 0 \quad \text{(for any closed loop)}.$$

Integration by parts generalized:

From identities introduced above we can infer relations between different integrals involving products of scalar and vector functions. In each case, one of the three integrals is amenable to one of the integral theorems.

$$\int_V d^3x\, f(\nabla\cdot\mathbf{G}) + \int_V d^3x\, \mathbf{G}\cdot(\nabla f)$$
$$= \int_V d^3x\, \nabla\cdot(f\mathbf{G}) \overset{\text{Gauss}}{=} \oint_S d\mathbf{a}\cdot(f\mathbf{G}).$$

$$\int_S d\mathbf{a}\cdot f(\nabla\times\mathbf{G}) + \int_S d\mathbf{a}\cdot[(\nabla f)\times\mathbf{G}]$$
$$= \int_S d\mathbf{a}\cdot[\nabla\times(f\mathbf{G})] \overset{\text{Stokes}}{=} \oint_C d\mathbf{l}\cdot(f\mathbf{G}).$$

$$\int_V d^3x(\nabla\times\mathbf{F})\cdot\mathbf{G} - \int_V d^3x\, \mathbf{F}\cdot(\nabla\times\mathbf{G})$$
$$= \int_V d^3x\, \nabla\cdot(\mathbf{F}\times\mathbf{G}) \overset{\text{Gauss}}{=} \oint_S d\mathbf{a}\cdot(\mathbf{F}\times\mathbf{G}).$$

Green's theorem:

Consider a region of space V bounded by a surface S. Two vector fields, $\phi\nabla\psi$ and $\psi\nabla\phi$, in that region are derived from a pair of scalar fields ϕ, ψ.

We apply an identity for the divergence of the vector fields thus constructed,

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi, \quad \nabla\cdot(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi.$$

Next we integrate the two identities over the region and use Gauss's theorem:

$$\int_V d^3x\big[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\big] = \oint_S da\,\phi\,\frac{\partial\psi}{\partial n}, \tag{1}$$
$$\int_V d^3x\big[\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi\big] = \oint_S da\,\psi\,\frac{\partial\phi}{\partial n}, \tag{2}$$

having set the stage for the surface integration as follows:

$$\phi\nabla\psi\cdot\hat{\mathbf{n}} = \phi\,\frac{\partial\psi}{\partial n}, \quad \psi\nabla\phi\cdot\hat{\mathbf{n}} = \psi\,\frac{\partial\phi}{\partial n}.$$

Green's theorem results from the difference of (1) and (2), which eliminates the gradient product terms:

$$\int_V d^3x\big[\phi\nabla^2\psi - \psi\nabla^2\phi\big] = \oint_S da\left[\phi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\phi}{\partial n}.\right]. \tag{3}$$

For some applications, Eq. (1) alone is useful by itself and also named Green's theorem. Green's theorem merely transforms a given problem.

Mathematical identities:

▷ First identity:

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{-1}{|\mathbf{x}-\mathbf{x}'|^2}\nabla|\mathbf{x}-\mathbf{x}'| = \frac{-1}{|\mathbf{x}-\mathbf{x}'|^2}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}.$$

The second transformation uses

$$\frac{\partial}{\partial x}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2} = \frac{x-x'}{|\mathbf{x}-\mathbf{x}'|}.$$

▷ Second identity:

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}').$$

(i) $\nabla^2(1/r) = 0$ for $r \neq 0$ readily verified in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[-\frac{1}{r^2}\right]\right) = 0.$$

(ii) volume integral of $\nabla^2(1/r)$ readily determined using Gauss's theorem:

$$\int_V d^3x\,\nabla\cdot\left(-\frac{\hat{\mathbf{r}}}{r^2}\right) = \oint_S\left(-\frac{\hat{\mathbf{r}}}{r^2}\right)\cdot d\mathbf{A} = -4\pi.$$

Helmholtz theorem:

Consider a vector function $\mathbf{H}(\mathbf{x})$ of which the divergence and curl are given:

$$\nabla \cdot \mathbf{H}(\mathbf{x}) = d(\mathbf{x}), \quad \nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{c}(\mathbf{x}).$$

The Helmholtz theorem states that these specifications uniquely determine the function $\mathbf{H}(\mathbf{x})$ under mild conditions.

Unique Decomposition: $\mathbf{H}(\mathbf{x}) = -\nabla\psi(\mathbf{x}) + \nabla \times \mathbf{A}(\mathbf{x})$,

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int d^3x' \frac{d(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \quad \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int d^3x' \frac{\mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$$

Here we show that the decomposition reproduces what is given.

- Show that $\nabla \cdot \mathbf{H}(\mathbf{x}) = d(\mathbf{x})$.
- Use identity: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.
- Consequence: $-\nabla^2\psi = d(\mathbf{x})$.
- Use identity: $\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.
- Application to integral expression:

  $$-\nabla^2\psi = -\frac{1}{4\pi}\int d^3x'\, \nabla^2\frac{d(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int d^3x'\, d(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') = d(\mathbf{x}).$$

- Show that $\nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{c}(\mathbf{x})$.
- Use identity: $\nabla \times (\nabla\psi) = 0$.
- Consequence: $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{c}(\mathbf{x})$.
- Use identity: $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$.
- Apply Laplacian to integral expression:

  $$-\nabla^2\mathbf{A} = -\frac{1}{4\pi}\int d^3x'\, \nabla^2\frac{\mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int d^3x'\, \mathbf{c}(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') = \mathbf{c}(\mathbf{x}).$$

- Show that $\nabla \cdot \mathbf{A} = 0$:

  $$4\pi\, \nabla \cdot \mathbf{A} = \int d^3x'\, \nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|} \cdot \mathbf{c}(\mathbf{x}') = -\int d^3x' \left[\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] \cdot \mathbf{c}(\mathbf{x}')$$
  $$= -\int d^3x' \left(\nabla' \cdot \left[\frac{\mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right] - \frac{\nabla' \cdot \mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right).$$

  The second integrand vanishes by construction: $\nabla \cdot \mathbf{c} \equiv 0$. The integral of the first term vanishes on account of Gauss's theorem (surface integral at infinity).