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Ballistic Particles in Helium Systems

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Ballistic motion of (quasi-)particles in helium systems with quantized and quasi-continuous spectrum is discussed under conditions when the mean free path is restricted by scattering on random surface inhomogeneities. The transport equation is derived for particles with arbitrary form of energy spectrum and without model assumptions on the structure of surface scattering operator. The results can be applied to quasiparticles in liquid helium systems with quadratic and linear spectra, spectrum with a gap, etc. The transport equation is relatively simple except for the case when the distance between quantized energy levels is comparable to the surface collision frequency. In three limiting cases the diffusion coefficient is calculated analytically for arbitrary correlations of surface inhomogeneities, and elsewhere - numerically for Gaussian correlations. The interwall correlation of surface inhomogeneities affects particle diffusion in a non-trivial way; sometimes, the effect of interwall correlations persists even in the quasiclassical limit.

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Low-temperature helium systems are perfect objects for experimental and theoretical study of ballistic motion of particles with various types of energy spectrum. At low temperatures, the mean free paths can be made sufficiently large so that boundary collisions become the dominant scattering mechanism. Boundary collisions can often be separated into two parts: scattering on surface-induced potential changes near the walls and scattering by (random) surface inhomogeneities. The latter problem is discussed below.

We consider (quasi-)particle diffusion in channel with random rough walls $x = \pm L/2 \mp \xi_{1,2}(y,z)$, $\xi_{1,2} \ll L$, $\langle \xi_{1,2} \rangle = 0$, with locally specular reflection in the absence of bulk scattering. The goal is to express the mean free path and transport coefficients via statistical and geometric characteristics

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of the wall profile, namely via the correlation function

$$\zeta_{ik}\left(|\mathbf{s}|\right) = \langle \xi_i(\mathbf{s}_1)\xi_k(\mathbf{s}_1+\mathbf{s})\rangle \equiv \int \xi_i(\mathbf{s}_1)\xi_k(\mathbf{s}_1+\mathbf{s})d\mathbf{s}_1, \tag{1}$$

The correlators ζ_{11} and ζ_{22} describe intrawall correlations of inhomogeneities, and $\zeta_{12} = \zeta_{21}$ - interwall correlations. Recently we developed a simple formalism[1] for exact mapping of transport problems with random boundaries onto problems with perfect boundaries and randomly distorted bulk. The formalism is based on Migdal-like canonical transformation that flattens the boundaries and, in the process, distorts the bulk. Here we report some new results and applications to helium systems. A more detailed account and literature review can be found elsewhere.[2]

Ballistic helium systems have two unique features. First, helium quasiparticles can have various types of energy spectra including quadratic spectrum (rotons and ^3He in $^3He - {}^4He$ mixtures), linear spectrum (phonons), anisotropic spectrum with gap, *etc.* Second, changing the width of ballistic channels and/or particle energy one can scan the whole range from ultra-quantum to quasi-continuous motion. We will express the transport coefficients via the arbitrary correlation functions, and supplement general expressions by computations for Gaussian correlations,

$$\zeta_{ik}\left(\mathbf{s}\right) = a_{ik}\exp\left(-s^2/2R_{ik}^2\right), \quad \zeta_{ik}\left(\mathbf{q}\right) = 2\pi a_{ik}R_{ik}^2\exp\left(-q^2R_{ik}^2/2\right) \tag{2}$$

To avoid parameter clutter, in numerical calculations we assume that all correlation lengths are the same, $R_{11} = R_{22} = R_{12} = R$, while the amplitudes a_{ik}, $\ell_{ik} = \sqrt{|a_{ik}|}\ell$ with some typical scale ℓ, may be different.

Earlier[1] we used the coordinate transformation

$$X = L[x - \left(\xi_2 - \xi_1\right)/2]/\left[L - \xi_1 - \xi_2\right] \tag{3}$$

which made the walls flat, $X = \pm L/2$. If one does not change coordinates y, z, the Jacobian J of such transformation $J \neq 1$, and the Hamiltonian $\widehat{H}$ in conjugate momentum variables $\{P_{x,y,z}\}$ can acquire non-Hermitian terms. To restore the volume, one has to supplement the transformation (3) by

$$Y = y\left(1+\gamma\left(\mathbf{s}\right)\right), \ \gamma\left(\mathbf{s}\right) = -\left(1/L\right)\int_0^1 \xi_+\left(\alpha\mathbf{s}\right)\alpha d\alpha, \ \xi_+ = \xi_1 + \xi_2, \tag{4}$$

(the same for z). It is possible to show[2] that transformation (4) is important only for anomalous quantum transport when the separation of minibands $\epsilon_{j\mathbf{q}} - \epsilon_{j'\mathbf{q}}$ for different states j, j' for finite motion in x-direction is comparable to the frequency of collision-induced transitions between minibands. Outside of this relatively narrow region (and always in quasiclassical conditions), this complication is irrelevant, and one should put $\gamma = 0$. In new momentum variables the Hamiltonian $\widehat{H} = p^2/2m = P^2/2m + \widehat{V}$ acquires the perturbation $\widehat{V}$,

$$\begin{aligned} V_{\mathbf{q}j,\mathbf{q}'j'} &= \frac{\delta_{jj'}}{mL}\left[\xi_+(\mathbf{q}'-\mathbf{q})\left(\frac{\pi j}{L}\right)^2 + \frac{1}{4}\left(q'^2 - q^2\right)(\mathbf{q}'+\mathbf{q})\cdot\frac{\partial\gamma(\mathbf{q}'-\mathbf{q})}{\partial(\mathbf{q}'-\mathbf{q})}\right] \\ &\quad -\frac{1-\delta_{jj'}}{mL}\frac{jj'\left(q'^2-q^2\right)}{j'^2-j^2}\left[\xi_1(\mathbf{q}'-\mathbf{q})(-1)^{j+j'} + \xi_2(\mathbf{q}'-\mathbf{q})\right], \\ \gamma(\mathbf{q}) &= -\frac{1}{L}\int_1^\infty \xi_+(\alpha\mathbf{q})\frac{d\alpha}{\alpha} \end{aligned}$$

The corresponding equation for an arbitrary energy spectrum is too cumbersome. However, outside the anomalous region, the collision operator retains its standard Waldmann-Snider form and contains the energy δ-function $\delta(\epsilon_{j\mathbf{q}} - \epsilon_{j'\mathbf{q}'})$ in the integrand. Then the expression for the matrix element can be simplified and obtains the following form for the arbitrary spectrum:

$$V_{\mathbf{q}j,\mathbf{q}'j'} = \frac{1}{L}\frac{jj'}{j^2-j'^2}\left[\xi_1(\mathbf{q}'-\mathbf{q}) + (-1)^{j+j'}\xi_2(\mathbf{q}'-\mathbf{q})\right](\epsilon_{j'\mathbf{q}} - \epsilon_{j\mathbf{q}'}) \tag{5}$$

The transport equation reduces to

$$\begin{aligned} \frac{dn_j}{dt} &= \frac{1}{L^2}\sum_{j'}\frac{j^2j'^2}{(j^2-j'^2)^2}\int\frac{d\mathbf{q}'}{2\pi}\left[\zeta_{11}+\zeta_{22}+2(-1)^{j+j'}\zeta_{12}\right] \\ &\quad\times(\epsilon_{j'\mathbf{q}}-\epsilon_{j\mathbf{q}'})^2\left(n_{j'}(\mathbf{q}') - n_j(\mathbf{q})\right)\delta(\epsilon_{j'\mathbf{q}'}-\epsilon_{j\mathbf{q}}) \end{aligned} \tag{6}$$

As it is clear from Eq.(6), the effect of the oscillating interwall term with $\zeta_{12}(\mathbf{q}'-\mathbf{q})$ is rather non-trivial.

The transport equation can be solved analytically for arbitrary type of correlation function ζ in three cases. In the first case, only one energy level is occupied $E = \epsilon_1(\mathbf{q}_1)$, and the transport equation becomes the same as usual $2D$ transport equation for impurity scattering with the combination

$$\Xi_{jj'} = \zeta_{11}(|\mathbf{q}_j - \mathbf{q}_{j'}|) + \zeta_{22}(|\mathbf{q}_j - \mathbf{q}_{j'}|) + 2(-1)^{j+j'}\zeta_{12}(|\mathbf{q}_j - \mathbf{q}_{j'}|)$$

playing the role of the transport cross-section (g is the spin factor),

$$D(E) = \frac{2gL^2}{\Xi_{11}^{(0)} - \Xi_{11}^{(1)}}\frac{(\partial\epsilon_{1\mathbf{q}_1}/\partial q_1)^3}{q_1\left(\partial\epsilon_{j\mathbf{q}_j}/\partial j\right)^2|_{j=1}}. \tag{7}$$

($\Xi_{11}^{(0,1)}$ are the angular harmonics of $\Xi_{11}(|\mathbf{q}_1 - \mathbf{q}_1'|)$ with $|\mathbf{q}_1 - \mathbf{q}_1'| = 2q_1\sin\theta/2$). The limiting case of longwave particles $q_jR \ll 1$ corresponds to quantum reflection when all the correlators $\zeta_{ik}(\mathbf{q}-\mathbf{q}')$ in the kernel of the integral equation can be replaced by the constants $\zeta_{ik}(0)$, and the diffusion coefficient of quasiparticles with the energy $E=\epsilon_{j\mathbf{q}_j}$ is equal to

$$D(E)=\frac{gL^2}{S}\sum_{j=1}^{S}\left(\frac{\partial\epsilon_{j\mathbf{q}_j}}{\partial q_j}\right)^2/\sum_{j'=1}^{S}\frac{q_{j'}\Xi_{jj'}(0)\,j^2j'^2\left(\epsilon_{j'\mathbf{q}_j}-\epsilon_{j\mathbf{q}_{j'}}\right)^2}{(j^2-j'^2)^2\left(\partial\epsilon_{j'\mathbf{q}_{j'}}/\partial q_{j'}\right)}. \tag{8}$$

In the third limit of ultra-narrow channels, $L\ll R$, the interband transitions are negligible in comparison with intraband scattering, and Eqs.(6) decouple,

$$D(E)=\sum_{j=1}^{S}\frac{2gL^2/S}{\Xi_{jj}^{(0)}-\Xi_{jj}^{(1)}}\frac{\left(\partial\epsilon_{j\mathbf{q}_j}/\partial q_j\right)^3}{j^2q_j\left(\partial\epsilon_{j\mathbf{q}_j}/\partial j\right)^2}. \tag{9}$$

For Gaussian correlations (2), Ξ is expressed via the hypergeometric function ${}_1F_1$:

$$\begin{aligned}\Xi_{jj'}(0) &= 2\pi\left(a_{11}+a_{22}+2(-1)^{j+j'}a_{12}\right)\ell^2R^2 \\ \Xi_{jj}^{(0)}-\Xi_{jj}^{(1)} &= 4\pi\left(a_{11}+a_{22}+2a_{12}\right)\ell^2R^2\,{}_1F_1\left(\frac{3}{2},2,-2q_j^2R^2\right)\end{aligned} \tag{10}$$

Examples of dependence of single-particle diffusion coefficient on $\alpha=R/\lambda(E)$ for two different situations are given in Figures 1, 2 for Gaussian correlation of inhomogeneities. Sharp singularities in both curves appear in the points where the number S of minibands $\epsilon_{j\mathbf{q}}$ accessible to a particle with energy E increases by 1 with increasing energy E.

Figure 1 illustrates the single-particle diffusion coefficient (8),

$$\begin{aligned}D &= \frac{gL^2}{m\ell^2}\Pi\left(\frac{R}{L},\alpha\right),\ q_j^2R^2=\alpha^2-\pi^2j^2R^2/L^2, \\ \Pi &= \frac{3L^4/\pi^2R^4}{S^2(S+1)(2S+1)}\sum_{j=1}^{S}\frac{1}{j^2}\frac{q_j^2R^2}{a_{11}+a_{22}+6(-1)^{j+S}a_{12}/(2S+1)},\end{aligned}$$

for (quasi-)particles with quadratic spectrum $\epsilon=-\Delta+p^2/2m$ as a function of energy $\alpha=R/\lambda(E)=R\sqrt{2mE}$, at $R/L=0.003$, $a_{11}=a_{22}=1$, and two values of interwall correlations, $a_{12}=0$ (solid line) and $a_{12}=0.7$ (dotted line). The contribution of interwall correlation is an oscillating function of the number S of minibands $\epsilon_{j\mathbf{q}}$ accessible to a particle with energy E and changes from destructive to constructive depending on whether S is even or odd. This is a general feature that can be observed for any energy spectrum at $R\ll\lambda\leq L$ for not very large values of S; at larger S, especially in the quasiclassical regime $S\gg1$, this effect of interwall correlations disappears.

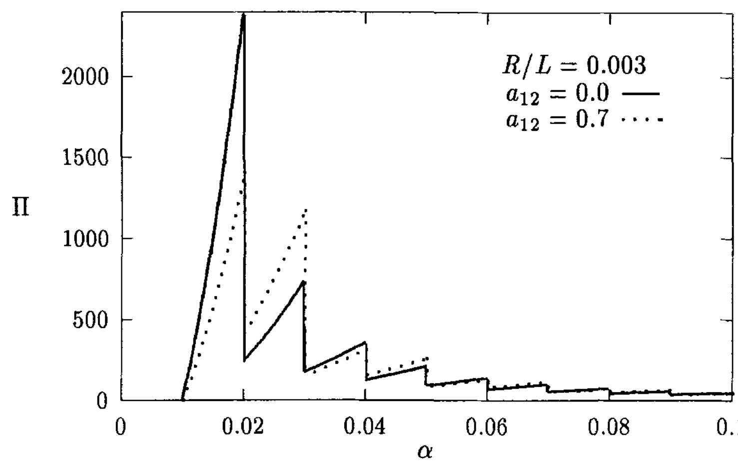

Fig. 1. Function $\Pi(\alpha)$, $\alpha = R/\lambda = R(2mE)^{1/2}$ for the single-particle diffusion coefficient $D = (gL^2/m\ell^2)\Pi(R/L, \alpha)$ at $R/L = 0.003$, $a_{11} = a_{22} = 1$, $a_{12} = 0$ (solid line) and $a_{12} = 0.7$ (dotted line)

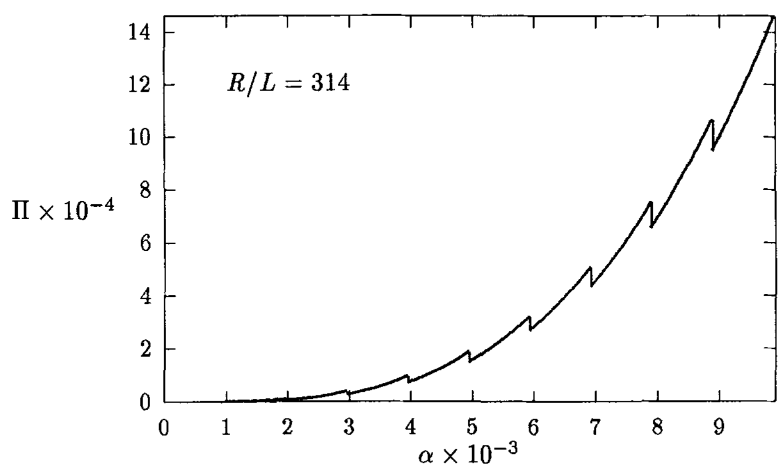

Fig. 2. Function $\Pi(\alpha)$, $\alpha = R/\lambda = R\omega/c$ for the single-phonon diffusion coefficient $D = (L^3c/\ell^2)\Pi(R/L, \alpha)$ at $R/L = 314$, $a_{11} = a_{22} = 1$ and $a_{12} = 0$

Figure 2 presents the single-phonon diffusion coefficient (9),

$$D = \frac{L^3 c}{\ell^2} \Pi \left(\frac{R}{L}, \alpha \right), \Pi = \frac{L^3/R^3}{2\pi^5 \alpha S} \frac{1}{a_{11} + a_{22} + 2a_{12}} \sum_{j=1}^{S} \frac{1}{j^4} \frac{q_j^2 R^2}{{}_1F_1 \left(\frac{3}{2}, 2, -2q_j^2 R^2 \right)}$$

$\epsilon = cp$, as a function of energy $\alpha = R/\lambda(E) = R\omega/c$ at $R/L = 314$, $a_{11} = a_{22} = 1$, and without interwall correlations, $a_{12} = 0$. In this case $j' = j$ and the whole effect of interwall correlations reduces to a trivial factor (10) $1/(a_{11} + a_{22} + 2a_{12})$.

In general, shapes of the curves depend more on parameters R/L and R/λ than on the type of the spectrum $\epsilon(\mathbf{p})$. The above expressions for diffusion coefficient $D(E)$ and mean free path $\mathcal{L} = 2D/v$ also provide one with the localization length $\mathcal{R}$ for particles with energy E[3,4]

$$\mathcal{R}(E) = \mathcal{L}(E) \exp[\pi m S(E) D(E)] \tag{11}$$

However, numerical estimates show that $2D$ localization (11) can be observed almost exclusively for low-energy particles for which only the first miniband $\epsilon_{1\mathbf{q}}$ is accessible, $S(E) = 1$, Eq.(7). At higher energies, the exponent (11) becomes too large.

The most straightforward application is for ^3He quasiparticles in ^3He-HeII mixtures with the spectrum $\epsilon_{j\mathbf{q}} = \Delta + (1/2m^*) \left(q^2 + (\pi j/L)^2 \right)$ and $q_j(E) = \left(2m^*(E - \Delta) - (\pi j/L)^2 \right)^2$. For these particles the above equations can be used almost without any modifications and restrictions. At very low temperatures, when the probability of inelastic phonon processes is low, the above equations can be used for helium phonons $\epsilon_{j\mathbf{q}} = c \left(q^2 + (\pi j/L)^2 \right)^{1/2}$ as well. For single-particle excitations in ^3He below the transition point the main restriction is presented by Andreev reflection. To account for these processes, the transport equation (6) should be re-written as a set of two coupled (sets of) equations for quasiparticles and quasiholes.

The above processes should define hydrodynamic flows in helium at ultralow temperatures and should strongly affect thermomechanical effect.

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